

# Locally compact abelian groups

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- Goals:
- What is a locally compact abelian group / the category  $\text{LCA}$ ?
  - Get an idea of Pontrjagin-van Kampen duality.
  - See an important structure theorem for  $\text{LCA}$  groups.

## §1 The category $\text{LCA}$

Def: A group  $G$  together with a topology  $\mathcal{T}$  on  $G$  is called topological group if the maps  $m: G \times G \rightarrow G$  and  $i: G \rightarrow G$  are continuous with respect to  $\mathcal{T}$ .

$(x, y) \mapsto x \cdot y$   
 $x \mapsto x^{-1}$

- Ex:
- $(\mathbb{R}, +)$  with the usual topology,  $(\mathbb{Q}^n, +) \subseteq (\mathbb{R}^n, +)$  subspace topology.
  - $\mathbb{R}/\mathbb{Z} \cong \mathbb{S}^1$  — u —
  - $\text{GL}_n(\mathbb{R})$  (subspace of  $\mathbb{R}^{n \times n}$ )
  - $\mathcal{C}(\mathbb{R}, \mathbb{S}^1)$  [with the compact-open topology]  $\otimes$
  - Any group with the discrete or indiscrete topology.

Def: A topological group  $(G, \mathcal{T})$  is called (locally) compact if  $\mathcal{T}$  is Hausdorff and (locally) compact.

- Ex:
- Any gp with the discrete topology.
  - $\mathbb{S}^1, \text{GL}_n \mathbb{R}, \mathbb{R}^n, \dots$
  - If  $A$  is abelian and locally compact, then  $\text{ctHom}(A, \mathbb{S}^1)$  is as well!
  - Lie groups
  - Not:  $\mathbb{Q}^n \subseteq \mathbb{R}^n, \mathbb{R}^{\mathbb{N}}, \dots$
- $\text{ctHom}(A, \mathbb{S}^1) \cong \mathbb{S}^1 \otimes \text{Hom}(A, \mathbb{S}^1)$   
 $\text{ctHom}(A, \mathbb{S}^1) := \{ f: A \rightarrow \mathbb{S}^1 \mid f(a) \cdot g(a) = 1 \}$

Def: We denote by  $\text{LCA}$  the category of locally compact abelian groups and their morphisms ( $\Rightarrow$  continuous group homomorphisms).

WARNING:  $\text{LCA}$  is not an abelian category!

It lacks, for example, cokernels:

$(\mathbb{Q}, \mathcal{T}_{\text{disc}}) \hookrightarrow (\mathbb{R}, \mathcal{T}_{\text{usual}})$  is a morphism

but: " $\mathbb{R}/\mathbb{Q}$ " is not locally compact!  $\nabla$

This is one of the main motivations for the seminar!

## §2 Pontrjagin-van Kampen duality

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A classical but powerful tool to study lca groups is Pontrjagin (-van Kampen) duality.

Def: Let  $A$  be an object of lca. We define

$\hat{A} := \text{chom}(A, \mathbb{S}^1) = \text{Mor}(A, \mathbb{S}^1)$ , equipped with the subspace topology of  $\mathcal{C}(A, \mathbb{S}^1)$ .

$\hat{A}$  is called the Pontrjagin dual of  $A$ .

Prop:  $\hat{A}$  is a locally compact abelian group.

See e.g.: "Locally compact groups, Stroppel", Thm. 20.5

Examples:

•  $\hat{\mathbb{R}} \cong \mathbb{R}$  → usual topology.

Let  $f: \mathbb{R} \rightarrow \mathbb{S}^1$  be a continuous homomorphism. We show:

$\exists! \omega \in \mathbb{R}: f(x) = \exp(i\omega x)$ .

Suppose  $f \neq \text{const}$ .

Then  $\ker(f) \subseteq \mathbb{R}$  is discrete (!)

~~Using the intermediate value theorem~~  $\exists \varepsilon > 0$  such that

$f|_{[0, \varepsilon]}$  is injective  $\rightarrow \exists \omega \in \mathbb{R}: f([0, \varepsilon]) = \{ \exp(i\omega \cdot s) \mid 0 \leq s \leq \varepsilon \}$   
↑ intermediate value thm.

$\rightarrow \forall \ell \in \mathbb{N} \exists! z \in f([0, \varepsilon]) : z^\ell = f(\varepsilon)$  ( $z = \exp(i\omega \varepsilon \cdot \frac{1}{\ell})$ )

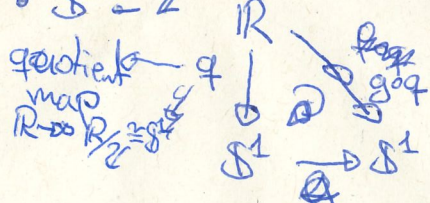
$\Rightarrow f(q \cdot \varepsilon) = \exp(i\omega q \varepsilon) \quad \forall q \in \mathbb{Q}$

$\Rightarrow f(t \cdot \varepsilon) = \exp(i\omega t \varepsilon) \quad \forall t \in \mathbb{R}$  (continuity of  $f$ )

$\Rightarrow$  algebraically:  $\hat{\mathbb{R}} \cong \mathbb{R}$ .

The topology agrees as well, but this requires more work.  
 (see e.g. Stroppel: Lemma 21.5)

•  $\hat{\mathbb{S}^1} \cong \mathbb{Z}$



$\Rightarrow \hat{\mathbb{S}^1} \cong_{\text{alg.}} \mathbb{Z}$

} more Details: Stroppel, lemma 21.6  
 Topology matches again...

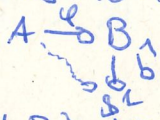
•  $\hat{\mathbb{Z}} \cong \mathbb{S}^1$

Since  $h_{\mathbb{Z}}: \mathbb{Z} \rightarrow \mathbb{S}(1)$  is uniquely determined by  $h(1)$ , and any choice makes a cont. hom.!

(Again, the topology matches, Stroppel, Lemma 21.7/21.8)

Proposition: Let  $A, B$  be LCA groups and  $\varphi \in \text{Mor}(A, B)$ .  
(=  $\text{Chom}(A, B)$ )

Then  $\hat{\varphi}: \hat{B} \rightarrow \hat{A}$  is a continuous homomorphism.  
 $\hat{b} \mapsto [\tau \mapsto \hat{b}(\varphi(\tau))]$



Pf: Stroppel, Lemma 20.2. (Idea: composition  $\mathcal{C}(A, B) \times \mathcal{C}(B, \mathbb{S}^1) \rightarrow \mathcal{C}(A, \mathbb{S}^1)$  is continuous)

Remark: We have seen, that " $\hat{\cdot}$ ": LCA  $\rightarrow$  LCA is a contravariant functor.

Proposition: Let  $A$  be an LCA group.

- (i)  $A$  discrete  $\Rightarrow \hat{A}$  compact
- (ii)  $A$  compact  $\Rightarrow \hat{A}$  discrete

Pf:

" $\Rightarrow$ "  $\mathcal{C}(A, \mathbb{S}^1) \cong \prod \mathbb{S}^1$  is compact.

$\varphi_{a,b}(\varphi): \mathcal{C}(A, \mathbb{S}^1) \rightarrow \mathbb{S}^1$   
 $\hat{a} = \bigcap_{a \in A} \varphi_{a,b}^{-1}(\{z \in \mathbb{S}^1 \mid \varphi(a,b)(z) = 1\})$

$\hat{A}$  is a closed subgroup of  $\mathcal{C}(A, \mathbb{S}^1)$  (!), hence compact

" $\Leftarrow$ " Set  $\Omega_{\frac{\pi}{2}} = \{f \in \hat{A} \mid f(A) \subseteq \{z \in \mathbb{S}^1 \mid |z-1| \leq \sqrt{2}\}\}$ .

The only subgp. contained in  $\Omega_{\frac{\pi}{2}}$  is  $\{1\}$  (!)

One can show:  $\Omega_{\frac{\pi}{2}}$  is a neighbourhood of  $1$  in  $\hat{A}$  (!)

$\Rightarrow \hat{A}$  is discrete.

More information: Stroppel, § 20.6, Theorem. □

Theorem [Pontrjagin-van Kampen duality]:

Let  $A$  be an LCA group. Then the map

$D_A: A \rightarrow \hat{\hat{A}}$  is an isomorphism of topological gps!  
 $x \mapsto [f \mapsto f(x)]$

Pf: Stroppel, § 22, especially Theorem 22.6.

Remark: This duality theorem is one of the most important tools in the study of LCA groups!

Q: Does this extend to a "bigger" category?

A: "No" (in this seminar). In general see e.g. "A survey on reflexivity of abelian topological gps", by Charac, Dikranjan, Martín-Peñador.

Examples/Problems:

- Consider  $(\mathbb{Q}, +) \subseteq (\mathbb{R}, +)$  with the subspace topology. no not loc. cpt. but abelian.

Topology and its Applications, 2012.

$$\text{ctHom}(\mathbb{Q}, \mathbb{S}^1) \cong \text{ctHom}(\mathbb{R}, \mathbb{S}^1)$$

(4)

$\mathbb{Q} \subseteq \mathbb{R}$  dense,  $f(\cdot)$  determines cont. Hom. uniquely.

$\Rightarrow (\mathbb{Q}, \mathcal{T}_{\text{subspace}}) \cong \widehat{\mathbb{R}} \cong \mathbb{R} \not\cong \mathbb{Q} \Rightarrow$  Thm fails.

### Cond(Ab):

- Since this is an abelian category, one can show:

If  $\text{Hom}(-, X)$  is "a duality ~~like~~ as Pontryagin-duality", then  $X$  is an injective object in  $\text{Cond}(\text{Ab})$

[more info: mathoverflow.net, question 440237]

- There are no non-trivial injective objects in  $\text{Cond}(\text{Ab})$ .

Attention: This is not true for  $\chi$ -condense abelian groups!

[more information: mathoverflow.net, question 352448]

### Cond(Ab) <sub>$\chi$</sub> :

-  $\mathbb{S}^1$  does not work in this setting, ~~Scholz~~

$$\underline{\text{Hom}}(A, B) \cong \underline{\text{ctHom}}(A, B) \quad (\text{Scholze, Proposition 4.2})$$

$\rightarrow$  Example with  $\mathbb{Q}, \mathcal{T}_{\text{subspace}}$ .

### §3 A structure theorem for LCA groups

Thm: Let  $G$  be an LCA group.

(i) Then  $G \cong E \oplus H$ , top. + algebraically,  $E \cong \mathbb{R}^n$  for some  $n$  and  $H$  an LCA gp. such that

(a)  $H$  contains a compact open subgroup. ( $\rightarrow$  Scholze: Thm 4.1 (i))

(b) Every compact subgroup of  $G$  is contained in  $H$ .

(ii) Every compactly generated LCA gp. is isomorphic to

$$\mathbb{R}^n \times K \times \mathbb{Z}^m, \quad K \text{ compact}, m, n \geq 0. \quad (\text{comp. gen. in gp. theory sense})$$

(iii) If  $G$  is connected, then  $G$  is the direct sum (algebraically and topologically) of a vector group and a unique

fin. dim.  $\rightarrow$  underlying ~~gp~~ ab. gp. of a topological vector space

maximal compact subgroup.

Pf: See [Hofmann-Morris, The structure of compact groups,] Theorem 7.57

$\rightarrow$  requires a lot of work!