Locally compact abelian groups

Goals:
- What is a locally compact abelian group / the category $\mathfrak{lca}$?
- Get an idea of Pontrjagin-van Kampen duality.
- See an important structure theorem for lca groups.

§1 The category $\mathfrak{lca}$

Def: A group G together with a topology $\mathcal{T}$ on G is called topological group if the maps $m: G\times G \to G$, $(x,y) \mapsto x\circ y$ and $\iota: G \to G$, $x \mapsto x^{-1}$ are continuous with respect to $\mathcal{T}$.

Ex:
- $(\mathbb{R}^n,+)$ with the usual topology, $(\mathbb{Q}^n,+) \leq (\mathbb{R}^n,+)$ subspace topology.
- $\mathbb{R}/\mathbb{Z} \cong \mathbb{S}^1$ —"—
- $GL_n(\mathbb{R})$ (subspace of $\mathbb{R}^{n\times n}$)
- $\mathcal{C}(\mathbb{R},\mathbb{S}^1)$ [with the compact-open topology] ⊛
- Any group with the discrete or indiscrete topology.

Def: A topological group $(G,\mathcal{T})$ is called (locally) compact if $\mathcal{T}$ is Hausdorff and (locally) compact.

Ex:
- Any gp with the discrete topology.
- $\mathbb{S}^1$, $GL_n\mathbb{R}$, $\mathbb{R}^n$, ...
- If A is abelian and locally compact, then $\mathrm{cHom}(A,\mathbb{S}^1)$ is as well!
  continuous homomorphisms
  ⊛ $(f+g)(a) := f(a)\cdot g(a)$.
- Lie groups
- Not: $\mathbb{Q}^n \leq \mathbb{R}^n$, $\mathbb{R}^{\mathbb{N}}$, ...

Def: We denote by $\mathfrak{lca}$ (LCA) the category of locally compact abelian groups and their morphisms ($\to$ continuous group homomorphisms).

WARNING: LCA is not an abelian category!

It lacks, for example, cokernels:

$(\mathbb{Q},\mathcal{T}_{disc}) \hookrightarrow (\mathbb{R},\mathcal{T}_{usual})$ is a morphism

but: "$\mathbb{R}/\mathbb{Q}$" is not locally compact!

This is one of the main motivations for the seminar!

§2 Pontrjagin-van Kampen duality

A classical but powerful tool to study lca groups is Pontrjagin (-van Kampen) duality.

Def: Let A be an ~~element~~ object of lca. We define

$\hat{A} := \mathrm{cHom}(A, \mathbb{S}^1) = \mathrm{Mor}(A, \mathbb{S}^1)$, equipped with the subspace topology of $\mathcal{C}(A, \mathbb{S}^1)$.

$\hat{A}$ is called the Pontrjagin dual of A.

Prop: $\hat{A}$ is a locally compact abelian group.

See e.g.: "Locally compact groups, Stroppel", Thm. 20.5

Examples:

- $\hat{\mathbb{R}} \cong \mathbb{R}$. → usual topology.

  Let $f: \mathbb{R} \to \mathbb{S}^1$ be a continuous homomorphism. We show:

  $\exists!\, \omega \in \mathbb{R}: f(x) = \exp(i\omega x)$.

  Suppose $f \neq$ const.

  Then $\ker(f) \subseteq \mathbb{R}$ is discrete (!)

  ~~Using the intermediate value theorem~~ $\exists\, \varepsilon > 0$ such that

  $f|_{[0,\varepsilon]}$ is injective $\leadsto \exists\, \omega \in \mathbb{R}: f([0,\varepsilon]) = \{\exp(i\omega \cdot s) \mid 0 \leq s \leq \varepsilon\}$

  intermediate value thm.

  $\leadsto \forall\, l \in \mathbb{N}\ \exists!\, z \in f([0,\varepsilon]): z^l = f(\varepsilon) \quad (z = \exp(i\omega\varepsilon \cdot \frac{1}{l}))$

  $\Rightarrow f(q \cdot \varepsilon) = \exp(i\omega q \varepsilon) \ \forall q \in \mathbb{Q}$

  $\Rightarrow f(t \cdot \varepsilon) = \exp(i\omega t \varepsilon) \ \forall t \in \mathbb{R}$ (continuity of f)

  $\Rightarrow$ algebraically: $\hat{\mathbb{R}} \cong \mathbb{R}$.

  The topology agrees as well, but this requires more work. (see e.g. Stroppel: Lemma 21.5)

- $\hat{\mathbb{S}}^1 \cong \mathbb{Z}$

  quotient map $\mathbb{R} \to \mathbb{R}/\mathbb{Z} \cong \mathbb{S}^1$

$$\left.\begin{tikzcd} \mathbb{R} \arrow[d, "q"'] \arrow[dr, "g \circ q"] & \\ \mathbb{S}^1 \arrow[r, "g"'] \arrow[ur, phantom, "\circlearrowleft"] & \mathbb{S}^1 \end{tikzcd}\right\} \Rightarrow \hat{\mathbb{S}}^1 \underset{\text{alg.}}{\cong} \mathbb{Z}$$

  } more Details: Stroppel, Lemma 21.6
  Topology matches again...

- $\hat{\mathbb{Z}} \cong \mathbb{S}^1$. Since $h: \mathbb{Z} \to \mathbb{S}(1)$ is uniquely determined by $h(1)$, and any choice makes a cont. hom.!

  (Again, the topology matches, Stroppel, Lemma 21.7 / 21.8)

<u>Proposition</u>: Let A, B be LCA groups and $\varphi \in \mathrm{Mor}(A,B)$.
$(= \mathrm{cHom}(A,B))$

Then $\hat{\varphi} : \hat{B} \to \hat{A}$ is a continuous homomorphism.
$\hat{b} \mapsto [t \mapsto \hat{b}(\varphi(t))]$

$A \xrightarrow{\varphi} B \xrightarrow{\hat{b}} S^1$

<u>Pf</u>: Stroppel, Lemma 20.2. (Idea: composition $\mathcal{C}(A,B) \times \mathcal{C}(B,S^1) \to \mathcal{C}(A,S^1)$ is continuous)

<u>Remark</u>: We have seen, that "^": LCA $\to$ LCA is a contravariant functor.

<u>Proposition</u>: Let A be an LCA group.
(i) A discrete $\Rightarrow \hat{A}$ compact
(ii) A compact $\Rightarrow \hat{A}$ discrete

<u>Pf</u>:

$\varphi_{a,b}(f) : \mathcal{C}(A,S^1) \to S^1$
$f \mapsto f(ab) f(a^{-1}) f(b^{-1})$

"$\Rightarrow$" $\mathcal{C}(A,S^1) \cong \prod_A S^1$ is compact. $\quad \hat{A} = \bigcap_{a,b \in A} \varphi_{a,b}^{-1}(\{1\})$

$\hat{A}$ is a closed subgroup of $\mathcal{C}(A,S^1)$ (!), hence compact

"$\Leftarrow$" Set $\Omega_{\frac{\pi}{2}} = \{ f \in \hat{A} \mid f(A) \subseteq \{ z \in S^1 \mid |z-1| \leq \sqrt{2} \} \}$.
The only subgp contained in $\Omega_{\frac{\pi}{2}}$ is $\{0\}$ (!)
One can show: $\Omega_{\frac{\pi}{2}}$ is a neighbourhood of 0 in $\hat{A}$ (!)
$\Rightarrow \hat{A}$ is discrete. □

<u>More information</u>: Stroppel, §20.6, Theorem.

<u>Theorem</u> [Pontrjagin-van Kampen duality]:
Let A be an LCA group. Then the map
$D_A : A \to \hat{\hat{A}}$ is an <u>isomorphism</u> of topological gps!
$x \mapsto [f \mapsto f(x)]$

<u>Pf</u>: Stroppel, §22, especially Theorem 22.6.

<u>Remark</u>: This duality theorem is one of the most important tools in the study of LCA groups!

<u>Q</u>: Does this extend to a "bigger" category?

<u>A</u>: "No" (in this seminar). In general see e.g. "A survey on reflexivity of abelian topological gps", by "Chasco, Dikranjan, Martín-Pejnador", Topology and its Applications, 2012.

<u>Examples + Problems</u>:

- Consider $(\mathbb{Q},+) \subseteq (\mathbb{R},+)$ with the subspace topology.
  $\leadsto$ not loc. cpt. but abelian.

$\mathrm{cHom}(\mathbb{Q}, \mathbb{S}^1) \cong \mathrm{cHom}(\mathbb{R}, \mathbb{S}^1)$

$\mathbb{Q} \subseteq \mathbb{R}$ dense, $f(1)$ determines cont. Hom. uniquely.

$\Rightarrow \widehat{\widehat{(\mathbb{Q}, \mathcal{T}_{subspace})}} \cong \hat{\hat{\mathbb{R}}} \cong \mathbb{R} \ncong \mathbb{Q} \rightsquigarrow$ Thm fails.

- Cond(Ab):
  - Since this is an abelian category, one can show: If $\mathrm{Hom}(-, X)$ is "a duality ~~like~~ as Pontryagin-duality", then X is an injective object in Cond(Ab) [more info: mathoverflow.net, question 440237]
  - There are no non-trivial injective objects in Cond(Ab).
    - Attention: This is not true for κ-condense abelian groups! [more information: mathoverflow.net, question 352448]
- $\mathrm{Cond(Ab)}_{\aleph}$:
  - $\underline{\mathbb{S}^1}$ does not work in this setting, ~~Schol~~ $\underline{\mathrm{Hom}}(\underline{A}, \underline{B}) \cong \underline{\mathrm{cHom}(A, B)}$ (Scholze, Proposition 4.2)

  $\rightsquigarrow$ Example with $\mathbb{Q}, \mathcal{T}_{subspace}$.

§3 A structure theorem for LCA groups

Thm: Let G be an LCA group.

(i) Then $G \cong E \oplus H$, top. + algebraically, $E \cong \mathbb{R}^n$ for some n and H an LCA gp. such that

ⓐ H contains a compact open subgroup. ($\rightsquigarrow$ Scholze: Thm 4.1 (i))

ⓑ Every compact subgroup of G is contained in H.

(ii) Every compactly generated LCA gp. is isomorphic to $\mathbb{R}^n \times K \times \mathbb{Z}^m$, K compact, $m, n \geq 0$. (comp. gen. in gp. theory sense)

(iii) If G is connected, then G is the direct sum (algebraically and topologically) of a $\underbrace{\text{vector group}}_{\text{fin. dim.}}$ and a unique maximal compact subgroup.

↳ underlying ~~gp~~ ab. gp. of a topological vector space

Pf: See [Hofmann-Morris, The structure of compact groups, Theorem 7.57]

$\rightsquigarrow$ requires a lot of work!