1 Abelian categories

Definition 1.1. Let $\mathcal{C}$ be a category. An **initial object** in $\mathcal{C}$ is an object $i \in \mathcal{C}$ such that there exists precisely one arrow $i \to c$ for each $c \in \mathcal{C}$. Dually, $t \in \mathcal{C}$ is called **terminal** if one has unique arrows $c \to t$ for each $c \in \mathcal{C}$.

An object that is both initial and terminal is called a **zero object**.

Remark. If $\mathcal{C}$ has a zero object z, it is unique up to isomorphism and for each $a, b \in \mathcal{C}$ there is a unique morphism

$$\begin{tikzcd} a \arrow[r] \arrow[rr, bend right, "0"'] & z \arrow[r] & b \end{tikzcd}$$

called the zero morphism from a to b.

Definition 1.2. Let $\mathcal{C}$ be a category with zero object and $a \xrightarrow{f} b$ be a morphism. The **(co)kernel** of f is the (co)equalizer of $f, 0 : a \to b$.

$$\begin{tikzcd} & c \arrow[d, dashed, "\exists!"'] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\ & \ker f \arrow[dl] \arrow[dr] & \\ a \arrow[rr, bend left, "f"] \arrow[rr, bend right, "0"'] & & b \end{tikzcd} \qquad \begin{tikzcd} & c & \\ & \operatorname{coker} f \arrow[u, dashed, "\exists!"] & \\ a \arrow[ur] \arrow[uur, bend left] \arrow[rr, bend left, "f"] \arrow[rr, bend right, "0"'] & & b \arrow[ul] \arrow[uul, bend right] \end{tikzcd}$$

Definition 1.3. A locally small category A is said to be **abelian** if the following conditions hold:

(i) A is **preadditive**: all of its hom-sets are abelian groups such that composition is bilinear, i.e. for $f_1, f_2 : a \to b$ and $g_1, g_2 : b \to c$,

$$(g_1 + g_2) \circ (f_1 + f_2) = (g_1 \circ f_1) + (g_1 \circ f_2) + (g_2 \circ f_1) + (g_2 \circ f_2);$$

(ii) A has all finite products and coproducts (and they are isomorphic);

(iii) A has a zero object;

(iv) Every morphism in A has a kernel and a cokernel;

(v) Every monomorphism (resp. epimorphism) is **normal**, meaning it is the kernel (resp. cokernel) of some morphism.

Example. Let A be an abelian category. A chain complex in A is a collection $(C_n, d_n)_{n\in\mathbb{Z}}$ of morphisms $d_n : C_n \to C_{n-1}$ such that $d_n \circ d_{n+1} = 0$.

One may form the category $\mathrm{Chn}(A)$ of chain complexes in A by looking at chain complexes as functors $\mathbb{Z} \to A$ and having morphisms be natural transformations between them:

$$\begin{tikzcd}
\dots \arrow[r] & C_{n+1} \arrow[r] \arrow[d, "f_{n-1}"] & C_n \arrow[r] \arrow[d, "f_n"] & C_{n-1} \arrow[r] \arrow[d, "f_{n+1}"] & \dots \\
\dots \arrow[r] & D_{n+1} \arrow[r] & D_n \arrow[r] & D_{n-1} \arrow[r] & \dots
\end{tikzcd}$$

$\mathrm{Chn}(A)$ is then also an abelian category.

Definition 1.4. Let A be an abelian category. The **image** of a morphism f is $\operatorname{im} f := \ker(\operatorname{coker} f)$. A sequence of morphisms $(\dots \to a_n \xrightarrow{f_n} a_{n+1} \to \dots)$ is called an **exact sequence** when $\operatorname{im} f_{n+1} \cong \ker f_n$, $n \in \mathbb{Z}$.

$$\begin{tikzcd}
a \arrow[rr, "f"] \arrow[dr, two heads] \arrow[ddr, bend right] & & b \\
& \operatorname{im} f \arrow[ur, hook] \arrow[d, dashed, "\exists!"'] & \\
& c \arrow[uur, hook, bend right] &
\end{tikzcd}$$

Definition 1.5. A non-empty category J is called **filtered** if

- given any $j, j' \in J$, there is a $k \in J$ with arrows $j \to k$, $j' \to k$;
- for every pair of parallel arrows $i \rightrightarrows j$, there exists $k \in J$ with a morphism $j \to k$ making $i \rightrightarrows j \to k$ commute.

Graphically, the following commutative diagrams always have solutions:

$$\begin{tikzcd}
j \arrow[dr, dashed] & & & & & \\
& k & & i \arrow[r, bend left] \arrow[r, bend right] & j \arrow[r, dashed] & k \\
j' \arrow[ur, dashed] & & & & &
\end{tikzcd}$$

Definition 1.6. Let A be an abelian category. One may require additional axioms on A, called the **Grothendieck axioms**:

(AB3+AB3*) A is (co)complete;

(AB4+AB4*) The product (resp. coproduct) of any family of epis (resp. monos) is an epi (resp. mono);

(AB5) Filtered colimits are exact, i.e. given exact sequences

$$\cdots \to a_j \to b_j \to c_j \to \dots$$

indexed by a filtered category J, the sequence

$$\cdots \to \operatorname*{colim}_{j\in J} a_j \to \operatorname*{colim}_{j\in J} b_j \to \operatorname*{colim}_{j\in J} c_j \to \dots$$

is also exact;

(AB6) If $(I_j)_{j\in J}$ is a family of filtered categories with functors $I_j \to A$, $i \mapsto a_i$, then there is an isomorphism

$$\operatorname*{colim}_{i_j\in I_j} \prod_{j\in J} a_{i_j} \xrightarrow{\cong} \prod_{j\in J} \operatorname*{colim}_{i_j\in I_j} a_{i_j}$$

2 Condensed abelian groups

Definition 2.1. A **condensed abelian group** is a contravariant functor $T : \mathrm{Prof} \to \mathrm{Ab}$ such that:

(S1) T turns finite coproducts into products i.e. given X, Y profinite, there is an isomorphism

$$T(X \sqcup Y) \to T(X) \times T(Y);$$

(S2) If the following is a pullback square with f surjective

$$\begin{tikzcd} X \times_B X \arrow[r, two heads, "p_1"] \arrow[d, two heads, "p_2"] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & B \end{tikzcd}$$

then T turns it into an equalizer diagram:

$$\begin{tikzcd} T(B) \arrow[r, "f^*"] & T(X) \arrow[r, shift left, "p_1^*"] \arrow[r, shift right, "p_2^*"'] & T(X \times_B X) \end{tikzcd}$$

Proposition 2.1. *The category of condensed abelian groups is equivalent to the category of contravariant functors* $\mathrm{ED} \to \mathrm{Set}$ *satisfying* (S1).

Lemma 2.1. *Limits in functor categories are computed pointwise. More precisely, let* C *be a category and*

$$\begin{aligned} F : J &\to \mathrm{Fun}\,(C^{\mathrm{op}}, \mathrm{Ab}) \\ j &\mapsto M_j \end{aligned}$$

be a diagram. Then the limit of F *exists and is given by*

$$(\lim_{j\in J} M_j)(c) \cong \lim_{j\in J}(M_j(c)).$$

Proof. This comes out easily once one uses the equivalence

$$\begin{aligned} \mathrm{Fun}(J, \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Ab})) &\xrightarrow{\simeq} \mathrm{Fun}(J \times C^{\mathrm{op}}, \mathrm{Ab}) \\ i \mapsto F_i &\longmapsto (j, c) \mapsto (F_j(c)) \end{aligned}$$

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Theorem 2.1. CondAb *is an abelian category satisfying all of Grothendieck's axioms.*

Proof. Let M_j, $j \in J$ be condensed abelian groups. Let's show that $\lim_{j\in J} M_j$ (seen as an object in $\mathrm{Fun}\,(C^{\mathrm{op}}, \mathrm{Ab})$) satisfies (S1):

$$\begin{aligned} (\lim_{j\in J} M_j)(S_1 \sqcup S_2) &= \lim_{j\in J}(M_j(S_1 \sqcup S_2)) \\ &= \lim_{j\in J}(M_j(S_1) \times M_j(S_2)) \\ &= \lim_{j\in J} M_j(S_1) \times \lim_{j\in J} M_j(S_2) \end{aligned}$$

From that, it is easy to see that CondAb is abelian: properties (ii), (iii), (iv) and (v), as well as (AB3), (AB4), (AB5) and (AB6), are all about (co)limits and are satisfied since Ab satisfies them (so the lemma guarantees it for CondAb as well). Also, given $M, N \in$ CondAb and natural transformations $\nu_1, \nu_2 : M \to N$, we may define $\nu_1 + \nu_2$ pointwise by $(\nu_1 + \nu_2)(S) = \nu_1(S) + \nu_2(S)$.

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Definition 2.2. A category C is said to be **generated** by $G \subseteq \operatorname{Obj} C$ if given any two distinct morphisms $a \rightrightarrows b$, there is an object $g \in G$ such that the following diagram does not commute:

$$\begin{tikzcd} g \arrow[r, dashed] & a \arrow[r, bend left] \arrow[r, bend right] & b \end{tikzcd}$$

Definition 2.3. An object $c \in C$ is said to be

- **projective** if given an epimorphism $\pi : e \twoheadrightarrow d$, any morphism $f : c \to d$ admits a lift $\tilde{f} : c \to e$:

$$\begin{tikzcd} & e \arrow[d, two heads, "\pi"] \\ c \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & d \end{tikzcd}$$

  This is equivalent to $\operatorname{Hom}(c, -)$ preserving epimorphisms (if C is abelian, this is also equivalent to $\operatorname{Hom}(c, -)$ being exact);

- **compact** if $\operatorname{Hom}(c, -)$ commutes with filtered colimits.

Lemma 2.2 (Yoneda Lemma). *If $F : C^{\mathrm{op}} \to \mathrm{Set}$ is a contravariant functor, then for each $c \in C$, there is a natural bijection*

$$\begin{aligned} \operatorname{Nat}(\operatorname{Hom}(-, c), F) &\longrightarrow F(c) \\ \nu &\longmapsto \nu_c(\mathrm{id}_c) \end{aligned}$$

Theorem 2.2 (Freyd's adjoint functor theorem). *If C is complete and $R : C \to D$ is a functor satisfying the "solution set condition", then R is continuous (resp. cocontinuous) $\Leftrightarrow$ R is a right (resp. left) adjoint.*

Theorem 2.3. CondAb *is generated by compact projective objects.*

Proof. The forgetful functor $U : \mathrm{CondAb} \to \mathrm{CondSet}$ is continuous, so it has a left adjoint $T \to \mathbb{Z}[T]$ by Freyd's adjoint functor theorem:

$$\begin{aligned} \mathbb{Z}[-] : \mathrm{CondSet} &\rightleftharpoons \mathrm{CondAb} : U \\ T &\mapsto \mathbb{Z}[T] \\ M &\leftarrow\!\shortmid M \end{aligned}$$

More explicitly, $\mathbb{Z}[T]$ is the so called **sheafification** of the functor

$$\begin{aligned} \mathbb{Z} \circ T : \mathrm{ED} &\to \mathrm{Ab} \\ S &\to \mathbb{Z}[T(S)] \end{aligned}$$

meaning it is the image of $\mathbb{Z} \circ T$ under the left adjoint of the inclusion $i : \mathrm{CondAb} \hookrightarrow \mathrm{Fun}(\mathrm{ED}^{\mathrm{op}}, \mathrm{Ab})$.

Let M be a condensed abelian group. Given an e.d. set S, we may form the condensed set $\underline{S} = C(-, S)$ and get natural isomorphisms

$$\begin{aligned} \mathrm{Hom}(\mathbb{Z}[\underline{S}], M) &\cong \mathrm{Hom}(\underline{S}, M) \\ &= \mathrm{Nat}(C(-, S), M) \\ &\cong M(S) \end{aligned}$$

Hence we may study the functor $\mathrm{Hom}(\mathbb{Z}(\underline{S}), -)$ by looking at $-(S)$ instead. Using that, let's show that the $\mathbb{Z}[\underline{S}]$ are projective compact generators:

• [**Compact**] Immediate, since $(\operatorname*{colim}_{j \in J} M_j)(S) \cong \operatorname*{colim}_{j \in J}(M_j(S))$.

• [**Projective**] In general, an arrow $f : a \to b$ is an epimorphism $\Leftrightarrow$ the following is a pushout square:

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "f"'] & b \arrow[d, "\mathrm{id}"] \\ b \arrow[r, "\mathrm{id}"] & b \end{tikzcd}$$

In our case, given an epimorphism $\nu : M \to N$, we get pushout squares

$$\begin{tikzcd} M \arrow[r, "\nu"] \arrow[d, "\nu"'] & N \arrow[d, "\mathrm{id}"] \\ N \arrow[r, "\mathrm{id}"] & N \end{tikzcd} \qquad \begin{tikzcd} M(S) \arrow[r, "\nu_S"] \arrow[d, "\nu_S"'] & N(S) \arrow[d, "\mathrm{id}"] \\ N(S) \arrow[r, "\mathrm{id}"] & N(S) \end{tikzcd}$$

so $-(S)$ preserves epimorphisms, i.e. $\mathbb{Z}(\underline{S})$ is projective.

• [**Generation**] Let $M \in \mathrm{CondAb}$ and consider all extremally disconnected sets S_j with an arrow $\nu_j : S_j \to M$, $j \in J$.

$$\begin{tikzcd} \mathbb{Z}[\underline{S_k}] \arrow[rr, "\nu_k"] \arrow[d] & & M \\ \bigoplus_j \mathbb{Z}[\underline{S_j}] \arrow[urr, dashed, "\bigoplus_j \nu_j"'] & & \end{tikzcd}$$

If we show that $\bigoplus \nu_j$ is an epimorphism, then given parallel arrows $M \rightrightarrows N$, the diagram $\bigoplus \mathbb{Z}[\underline{S_j}] \overset{\bigoplus \nu_j}{\to} M \rightrightarrows N$ will not commute, and consequently $\mathbb{Z}[\underline{S_k}] \overset{\nu_k}{\to} M \rightrightarrows N$ will not commute for some $k \in J$.

Define the poset of **subobjects** of M as the set of isomorphism classes of monomorphisms into M with partial order defined by $M' \leq M''$ iff there is a factorization

$$\begin{tikzcd} M' \arrow[r, hook] \arrow[d, dashed] & M \\ M'' \arrow[ur, hook] & \end{tikzcd}$$

This has a subposet given by monomorphisms $M' \hookrightarrow M$ admitting an epimorphism $\bigoplus \mathbb{Z}[\underline{S_j}] \twoheadrightarrow M'$, and, given a chain $\cdots \leq M'_n \leq M'_{n+1} \leq \ldots$, the image of $\bigoplus M'_n \to M$ is an upper bound. So Zorn's lemma guarantees the existence of a maximal subobject $M' \hookrightarrow M$. We want to show that $M \cong M'$.

Suppose that $M/M' := \operatorname{coker}(M' \hookrightarrow M) \neq 0$. Then $\operatorname{Hom}(\mathbb{Z}[\underline{S}], M/M') \neq 0$ for some S, so there is a non-zero map $f : \mathbb{Z}[\underline{S}] \to M/M'$ lifting to a map $\tilde{f} : \mathbb{Z}[\underline{S}] \to M$:

$$\begin{tikzcd} & M \arrow[d, two heads] \\ \mathbb{Z}[\underline{S}] \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & M/M' \end{tikzcd}$$

So $\operatorname{im} \tilde{f} \neq 0$ and, given an epimorphism $\nu' : \bigoplus \mathbb{Z}[\underline{S_j}] \twoheadrightarrow M'$, there is a factorization

$$\begin{tikzcd} \bigoplus_j \mathbb{Z}[\underline{S_j}] \oplus \mathbb{Z}[\underline{S}] \arrow[r, two heads, "\nu' + \tilde{f}"] & M' \oplus \operatorname{im} \tilde{f} \arrow[r, hook] & M \\ & M' \arrow[u, hook] \arrow[ur, hook] & \end{tikzcd}$$

So $M' < M' \oplus \operatorname{im} \tilde{f}$, contradicting the maximality of M'. Therefore $M/M' = 0$, i.e. $M' \cong M$. ■

Definition 2.4. An abelian category with a generator satisfying (AB3) and (AB5) is called a **Grothendieck category**.