Basic Category Theory.

Categories.

A **category** consists of

- a class of objects $\mathrm{ob}(C)$,
- a class of morphisms or arrows $\hom(C)$,
- a domain or source object class function $\mathrm{dom} : \hom(C) \to \mathrm{ob}(C)$,
- a codomain or target object class function $\mathrm{cod} : \hom(C) \to \mathrm{ob}(C)$,
- for every three objects a, b and c, a binary operation $\hom(a, b) \times \hom(b, c) \to \hom(a, c)$ called composition of morphisms. We will denote the composition of $f : a \to b$ and $g : b \to c$ as $g \circ f$ or gf.

such that the following axioms hold:

- (associativity) if $f : a \to b, g : b \to c$ and $h : c \to d$ then $h \circ (g \circ f) = (h \circ g) \circ f$, and
- (identity) for every object x, there exists a morphism $\mathrm{id}_x : x \to x$ called the identity morphism for x, such that every morphism $f : a \to x$ satisfies $\mathrm{id}_x \circ f = f$, and every morphism $g : x \to b$ satisfies $g \circ \mathrm{id}_x = g$.

Examples:

- $\mathrm{ob}(C)$ = sets, $\hom(C)$ = maps between sets.
- $\mathrm{ob}(C)$ = groups, $\hom(C)$ = group homomorphisms.
- $\mathrm{ob}(C)$ = topological spaces,
  $\hom(C)$ = continuous functions between topological spaces.

A category is called **small** if the class of objects and the class of morphisms are sets and **large** otherwise.

A **subcategory** C' of C is a category, such that:

- Objects of C' are objects in C
- For an ordered pair (X', Y') of objects in C' $\hom_{C'}(X', Y') \subset \hom_C(X', Y')$.
- For morphisms $f' \in \hom(Y', Z')$ and $f' \in \hom_{C'}(Y', Z')$ the composition in C' is the same as in C.

For a category C we define the **opposite category** C^{op} as follows:

- objects of C^{op} = objects of C
- morphisms of C^{op}: $\hom_{C^{\mathrm{op}}}(X, Y) = \hom_C(Y, X)$

Morphisms.

For a morphism $f \in \hom(X, Y)$ a morphism $g \in \hom(Y, X)$ is called a **left inverse** of f if $g \circ f = \mathrm{id}_X$. (**Right inverse** analogously.)

If g is a left and right inverse for f then f is called an **isomorphism**.

A category is called **skeletal** when any two isomorphic objects are identical; i.e. when the category is its own skeletal.

A morphism $f : a \to b$ is called

- a **monomorphism** if it is left-cancellable, i.e. $f \circ g_1 = f \circ g_2 \Rightarrow g_1 = g_2$ for all morphisms $g_1, g_2 : x \to a$.
- an **epimorphism** if it is right-cancellable, i.e. $g_1 \circ f = g_2 \circ f \Rightarrow g_1 = g_2$ for all morphisms $g_1, g_2 : b \to x$.

Remark: Epimorphisms are not necessarily surjective. Consider the inclusion $\iota : \mathbb{Z} \hookrightarrow \mathbb{Q}$. This is an epimorphism in Rings. Suppose $g, h : \mathbb{Q} \to A$ ring homomorphisms agreeing on $\mathbb{Z}$. Then $g = h$, because: For any $n \in \mathbb{Z}$ we have $g(n) = h(n)$, for $m \in \mathbb{Z}\backslash\{0\}$ we have

$$g(1/m) = g(m)^{-1} = h(m)^{-1} = h(1/m).$$

So $g \circ \iota = h \circ \iota \Rightarrow g = h$.

An object X of a category C is called **initial** if $\hom(X, Y)$ consists of exactly one element for every object Y.

An object Y of a category C is called **terminal** if $\hom(X, Y)$ consists of exactly one element for every object X.

Functors.

Let C, D be categories. A **(covariant) functor** $F\colon C \to D$ is a mapping that

- associates each object X in C to an object $F(X)$ in D.
- associates each morphism $f : X \to Y$ in C to a morphism $F(f) : F(X) \to F(Y)$ in D such that:
  - $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$
  - $F(g \circ f) = F(g) \circ F(f)$ for all morphisms $f : X \to Y,\ g : Y \to Z$ in C.

Contravariant functors $F : A \to B$ are covariant functors $F : A^{\mathrm{op}} \to B$.

A functor $S : C \to D$ is an **isomorphism of categories** when there is a functor $T : D \to C$ such that $ST \simeq \mathrm{id}_D$ and $TS \simeq \mathrm{id}_C$.

Let C, D be categories, $F, G : C \to D$ functors. A **natural transformation** $\eta : F \to G$ is a mapping that maps every object $X \in C$ to a morphism $\eta_X : F(X) \to G(X)$ such that for every morphism $f : X \to Y$ in C the following diagram commutes:

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle \eta_X}\big\downarrow & & {\scriptstyle \eta_Y}\big\downarrow \\ G(X) & \xrightarrow{G(f)} & (GY) \end{array}$$

Example: In Groups: $F = \mathrm{id}_{\mathrm{Grps}},\ G = (\_)^{\mathrm{ab}}$ (Abelianization), $q_H : H \to H^{\mathrm{ab}} = H/[H,H]$.

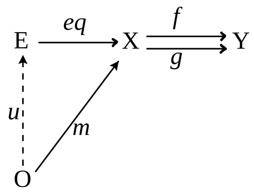
A functor $S : C \to D$ is an **equivalence of categories** when there is a functor $T : D \to C$ and natural isomorphisms $ST \cong \mathrm{id}_D$ and $TS \cong \mathrm{id}_C$. In this case T is also an equivalence of categories.

Examples:

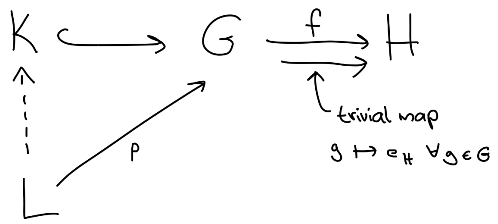
- A category is equivalent to any one of its skeleta.
- $\{\mathbb{R}^n\}_{n\in\mathbb{N}}$ is a skeletal subcategory for finite dimensional real vector spaces. (Let V be an n-dim real vector space. For any basis $v_1, \ldots, v_n \in V$ each element of V is uniquely expressable as $a_1v_1 + \cdots + a_nv_n$ for some $a_1, \ldots, a_n \in \mathbb{R}$. One gets isomorphisms $(a_1, \ldots, a_n) \mapsto (a_1v_1 + \cdots + a_nv_n)$.)

Limits.

The **equaliser** consists of an object E and a morphism $\text{eq} : E \to X$ satisfying $f \circ \text{eq} = g \circ \text{eq}$ such that, given any object O and morphism $m : O \to X$, if $f \circ m = g \circ m$ then there exists a unique morphism $u : O \to E$ such that $\text{eq} \circ u = m$.



Example: In Groups: Let G, H, L be groups, $f : G \to H$ and $K = \ker(f)$.



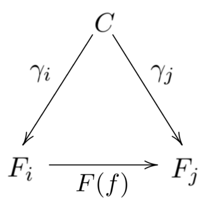
(Equalisers generalise kernels.)

Let I be a small category and C a category. Then we define a **functor category** or **diagram category** C^I as follows:

- objects: functors from I to C
- morphisms: natural transformations of such functors

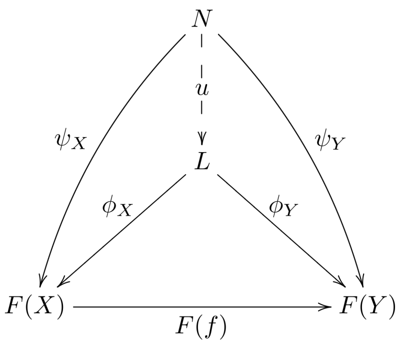
For all objects $c \in C$ there exists a constant functor : $\underline{c} : I \to C$ with $\underline{c}(i) = c$ for all objects $i \in I$, $\underline{c}(f) = \text{id}_C$ for all arrows $f \in I$.

A **cone** over a diagram $F \in C^I$ is an object C and morphisms $\gamma_i : C \to F_i$ for all objects $i \in I$ such that for each $(f : i \to j) \in I$ the following triangle commutes

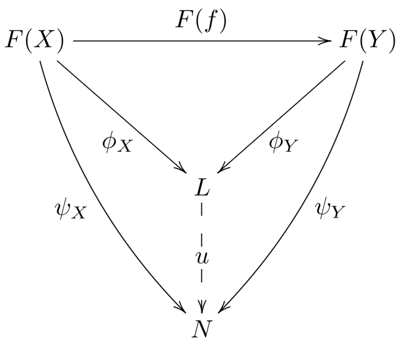


A cone over F can be seen as a morphism $\underline{c} \to f$ in C^I.

A **limit of the diagram** $F : J \to C$ is a cone (L, ϕ) to F such that for every other cone (N, ψ) to F there exists a unique morphism $u : N \to L$ such that $\phi_X \circ u = \psi_X$ for all X in J.



Cocones and **colimits** are the dual notions of cones and limits. We obtain them by inverting arrows. A cocone can be seen as a natural transformation $f \to \underline{c}$ in C^I.



Equalisers (pullbacks, pushouts, ...) are examples for limits. A limit of $F : I \to C$ is a terminal object in Cone(F).

Homsets preserve limits. We have:

$$\begin{aligned}\hom_C(X, \lim F_i) \leftrightarrow \mathrm{Cone}_I(X, F_i) \leftrightarrow \mathrm{Cone}_{\mathrm{Sets}}(\mathrm{pt}, \hom_C(X, F_i))\\ \leftrightarrow \lim \hom_C(X, F_i)\end{aligned}$$

Adjoint functors.

Let $F : C \to D$ and $G : D \to C$ be functors. F and G are called **adjoint**, if there is a bijection $\forall c, d \in \mathrm{ob}(C)$ between $\hom_D(Fc, d)$ and $\hom_C(c, Gd)$ that is natural in c and d.

Naturality in c means that for each $(f \colon c' \to c) \in C$, the following diagram commutes:

$$\begin{array}{ccc} \hom_D(Fc,d) & \xrightarrow{\eta_{c,d}} & \hom_C(c,Gd) \\ \big\downarrow{\scriptstyle -\circ Ff} & & \big\downarrow{\scriptstyle -\circ f} \\ \hom_D(Fc',d) & \xrightarrow{\eta_{c',d}} & \hom_C(c',Gd). \end{array}$$

Naturality in d means that for each $(g : d \to d') \in D$, the following diagram commutes:

$$\begin{array}{ccc} \hom_D(Fc,d) & \xrightarrow{\eta_{c,d}} & \hom_C(c,Gd) \\ \big\downarrow{\scriptstyle g\circ -} & & \big\downarrow{\scriptstyle Gg\circ -} \\ \hom_D(Fc,d') & \xrightarrow{\eta_{c',d}} & \hom_C(c,Gd'). \end{array}$$

Example: F : Sets $\to$ Grps maps a set to the free group over that set, U : Grps $\to$ Sets maps a group to the set of group elements (it is called the forget functor, because it *forgets* about the additional structure a group has).

- Set maps: $X \to UG$
- Group homomorphisms: $FX \to G$

Remark: Adjunction generalizes the notion of equivalence, inducing natural transformations $\eta\colon \mathrm{id}_C \to GF$ and $\eta\colon FG \to \mathrm{id}_D$, called the *unit* and *counit* of the adjunction, respectively, which need not be natural isomorphisms. An adjunction can alternatively be axiomatized in terms of two functors equipped with a unit and counit satisfying certain identities.

Example: Čech–Stone compactification:

Let X be a Tychonov space and let $\iota_X : X \to \prod_{C(X,[0,1])}[0,1]$ be a map with $\iota_X(p) = (\varphi(p))_{\varphi \in C(X,[0,1])}$. The **Čech–Stone compactification** is $\beta X := \overline{\iota_X(X)}$.

If X, Y are Tychonov spaces and $\xi : X \to Y$ is a continuous map, then there exists a unique continuous map $\beta\xi : \beta X \to \beta Y$, such that $\iota_Y \circ \xi = \beta\xi \circ \iota_X$ (such that the following diagram commutes:)

$$\begin{array}{ccc} X & \xrightarrow{\xi} & Y \\ {\scriptstyle \iota_X}\downarrow & & {\scriptstyle \iota_Y}\downarrow \\ \beta X & \xrightarrow{\beta\xi} & \beta Y \end{array}$$

The Čech–Stone compactification is a left-adjoint functor for the inclusion map $G : \mathrm{CptHaus} \hookrightarrow \mathrm{Tyc}$. We have:

$$\hom_{\mathrm{Tyc}}(X, GK) \leftrightarrow \hom_{\mathrm{CptHaus}}(\beta X, K)$$

One can see that Čech–Stone compactification is a functor by using these diagrams:

$$\begin{array}{ccccc} X & \xrightarrow{g} & Y & \xrightarrow{f} & Z \\ {\scriptstyle \iota_X}\downarrow & & {\scriptstyle \iota_Y}\downarrow & & {\scriptstyle \iota_Z}\downarrow \\ \beta X & \xrightarrow{\beta g} & \beta Y & \xrightarrow{\beta f} & \beta Z \end{array} \qquad \beta X \xrightarrow{\beta(f \circ g)} \beta Z$$

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}_X} & X \\ {\scriptstyle \iota_X}\downarrow & & {\scriptstyle \iota_X}\downarrow \\ \beta X & \xrightarrow{\mathrm{id}_{\beta X} = \beta\, \mathrm{id}_X} & \beta X \end{array}$$

Fact: Right adjoints preserve limits. Left adjoints preserve colimits.

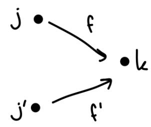
A subcategory A of B is called **reflective** in B when the inclusion functor $K : A \to B$ has a left adjoint $F : B \to A$.

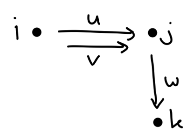
Example: The Čech–Stone compactification shows that compact Hausdorff spaces are a reflective subcategory of the category of Tychonov spaces. Abelianization shows that abelian groups are a reflective subcategory of the category of groups.

Filtered (co)limits.

A category J is **filtered** when

- it is not empty,
- for every two objects j and j' in J there exists an object k and two arrows $f : j \to k$ and $f' : j' \to k$ in J,
- for every two parallel arrows $u, v : i \to j$ in J, there exists an object k and an arrow $w : j \to k$ such that $wu = wv$.





A **filtered colimit** is a colimit of a functor $F : J \to C$ where J is a filtered category. Equivalently every finite diagram in a filtered category C admits a cocone under it.

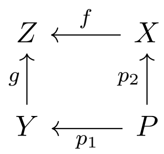
Filtered colimits commute with finite limits in some categories (Sets, Top, Grps, Rings). For Sets see Theorem IX.2.1, p. 215 in Mac Lane's *Categories for the Working Mathematician.* To see this is also true in Grps, Rings, and Top requires knowledge of how filtered colimits are computed in these categories (they are created by the forgetful functor to Sets; see Prop. IX.1.2 for Grps).

More generally, let κ be a regular cardinal. We say a category C is κ-filtered if every diagram $I \to C$ from a category I with fewer than κ-many arrows admits a cocone.

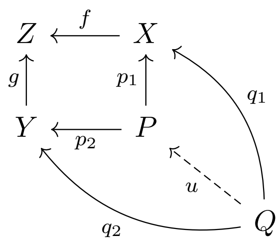
In Sets, κ-filtered colimits commute with κ-small limits; the previous statement is the case $\kappa = \aleph_0$.

Appendix.

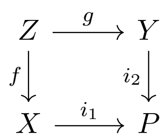
A **pullback of morphisms** f **and** g consists of an object P and two morphisms $p_1 : P \to X$ and $p_2 : P \to Y$ such that the following diagram commutes



and such that the pullback (P, p_1, p_2) is universal with respect to this diagram. That is for any other (Q, q_1, q_2) where $q_1 : Q \to X$ and $q_2 : Q \to Y$ are morphisms with $f \circ q_1 = g \circ q_2$ there must exist a unique $u : Q \to P$ such that $p_1 \circ u = q_1$ and $p_2 \circ u = q_2$.



A **pushout of the morphisms** f **and** g consists of an object P and two morphisms $i_1 : X \to P$ and $i_2 : Y \to P$ such that the following diagram commutes



and such that the pushout (P, i_1, i_2) is universal with respect to this diagram. That is for any other (Q, j_1, j_2) where $j_1 : X \to Q$ and $j_2 : Y \to Q$ are morphisms with $j_1 \circ f = j_2 \circ g$ there must exist a unique $u : Q \to P$ such that $u \circ i_1 = j_1$ and $u \circ i_2 = j_2$.