18.04.'23

Compact spaces II

Reminder:

(0) A top. Hausdorff space X is called <u>extremally disconnected</u> (e.d.) if for every open $U \subseteq X$ the closure $\overline{U}$ is open in X.

$\iff$ $\forall$ pair $U, V \subseteq X$ open, $U \cap V = \emptyset$ is $\overline{U} \cap \overline{V} = \emptyset$.

(1) Universal property of the Čech-Stone-Cpt.

X top. Hausdorff space

continuous   cpt

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "i"', "\text{dense}"] & K \\ \beta X \arrow[ur, dashed, "\exists! \text{ continuous } \beta h"'] \arrow[u, phantom, shift left=6ex, "\circlearrowright"] & \end{tikzcd}$$

$\forall K$ cpt and $h: X \longrightarrow K$ continuous $\exists!$ continuous $\beta h: \beta X \longrightarrow K$ with $\beta h \circ i = h$

<u>Fact 1:</u> βX is e.d. iff X is e.d.

(2) an object X of a category $\mathcal{C}$ is called <u>projective</u> if

$$\begin{tikzcd} & Y \arrow[d, two heads, "g"] \\ X \arrow[ur, dashed, "\exists \overline{f}"] \arrow[r, "f"'] \arrow[ur, phantom, shift right=3ex, "\circlearrowright"] & Z \end{tikzcd}$$

$\forall Y, Z \in \mathcal{C}$ and $g: Y \twoheadrightarrow Z$ epim. and $f: X \longrightarrow Z$ morph. $\exists \overline{f}: X \longrightarrow Y$ morph with $g \circ \overline{f} = f$

(3) X top. space, $Y \subseteq X$ is called <u>retract</u> of X, if $\exists r: X \longrightarrow Y$ continuous, s.t. $r(y) = y$ $\forall y \in Y$.

<u>Fact 2:</u> A retract of an e.d. space is e.d.

proof: Let $U, V \subseteq Y$ open, $U \cap V = \emptyset$

$\rightsquigarrow r^{-1}(U), r^{-1}(V) \subseteq X$ open and disjoined

$\overset{X \text{ e.d.}}{\rightsquigarrow}$ $\overline{f^{-1}(U)} \cap \overline{f^{-1}(V)} = \emptyset$

$\overline{f^{-1}(U)} \supseteq \overline{U}$, $\overline{f^{-1}(V)} \supseteq \overline{V}$ (closure in Y) $\rightsquigarrow \overline{U} \cap \overline{V} = \emptyset$ $\square$

Lemma A X cpt, then there ex. a discrete space D and a continuous surj. map $\beta D \twoheadrightarrow X$.

proof: Let $D = X^\delta$

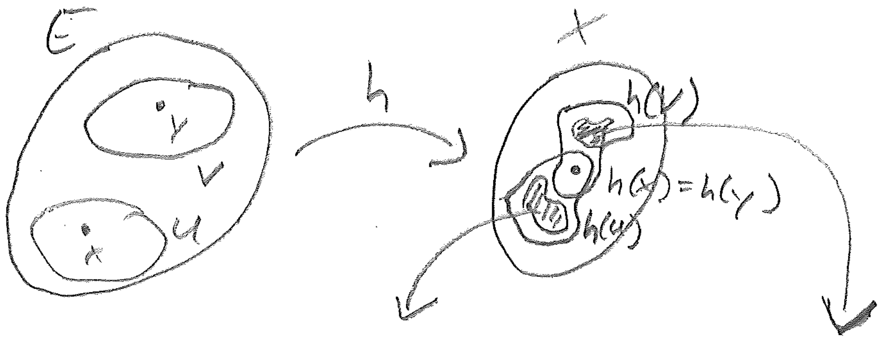
$$\begin{tikzcd} D \arrow[r, two heads, "id"] \arrow[d, hook] & X \\ \beta D \arrow[ur, dashed, two heads, "\beta id"'] & \end{tikzcd}$$ $\square$

Lemma B X e.d. cpt space, E cpt, $h: E \twoheadrightarrow X$ cont. s.t. $\forall E' \subsetneq E$ closed : $h(E') \neq X$.

Then h is a homeomorphism.

proof: (Have to show injectivity)

A: $x \neq y \in E$, $h(x) = h(y)$. $U, V \subseteq E$ open, $U \cap V = \emptyset$, $x \in U$, $y \in V$

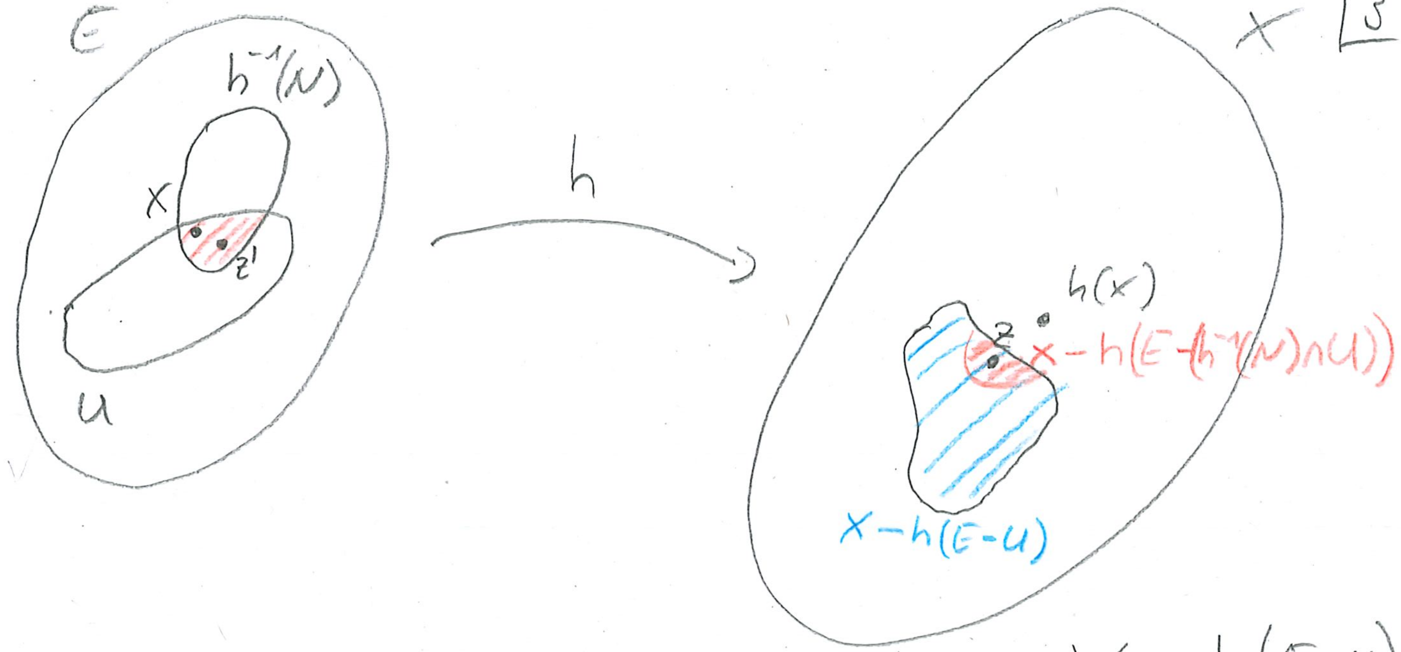


$X - h(E-U)$, $X - h(E-V)$ open + disjoined in X

$\overset{X \text{ e.d.}}{\rightsquigarrow}$ $S := \overline{X - h(E-U)} \cap \overline{X - h(E-V)} = \emptyset$

But: $h(x) = h(y) \in S$, because:

Let $N \subseteq X$ open, $h(x) \in N$



$$\exists z \in X - h\big(\underbrace{E - (\underbrace{h^{-1}(N) \cap U}_{\ni x})}_{\neq X}\big) \subseteq X - h(E-U)$$

$$\exists z' \in E : h(z') = z \quad \rightsquigarrow \quad z' \in h^{-1}(N) \cap U$$

$$\rightsquigarrow \quad z \in N \cap (X - h(E-U))$$

$$\rightsquigarrow \quad h(x) \in \overline{X - h(E-U)} \quad ⚡$$

$$\overset{\shortparallel}{h(y)} \in \overline{X - h(E-V)} \qquad \square$$

<u>Lemma C</u> X, K cpt, $\pi : K \twoheadrightarrow X$ continuous.
Then $\exists E \subseteq K$ cpt, s.t. $\pi(E) = X$ and $\forall E' \subsetneq E$ closed $\pi(E') \neq X$.

proof:
$$M = \{ E \subseteq K \text{ closed} \mid \pi(E) = X \} \neq \emptyset$$

$$E \leq E' \quad :\Leftrightarrow \quad E' \subseteq E$$

for $x \in X$: $\emptyset \neq \pi^{-1}(x) \cap E_i$ cpt $\Downarrow$

$\rightsquigarrow$ Every chain $(E_i)_{i \in N}$ has an upper bound $\bigcap_{i \in N} E_i \in M$

$\xRightarrow{\text{Lemma of Zorn}}$ $\exists$ $E \in M$ max. $\Leftrightarrow$ $\pi(E) = X$ and for $E' \subsetneq E$ closed $\pi(E') \neq X$. $\square$

Thm For X cpt, the following are equivalent:

(i) X is e. d.

(ii) X is projective in the category of cpt. spaces

(iii) X is a retract of βD for some discrete D.

proof: „(i) $\Rightarrow$ (ii)"

$$\begin{tikzcd} & & Y \arrow[d, two heads, "g"] \\ X \arrow[urr, dashed, "\text{show } \exists \bar{f}"] \arrow[rr, "f"'] & & Z \end{tikzcd}$$

$K = \{(x,y) \in X \times Y \mid f(x) = g(y)\}$
cpt in $X \times Y$

$\pi_X(K) = X$, where $\pi_X : X \times Y \longrightarrow X$ can. proj.

Lem C $\rightsquigarrow$ $\exists E \subseteq K$ closed s.t. $\pi_X(E) = X$ and $\forall E' \subsetneq E$ closed:
$\pi_X(E') \neq X$

Lem B $\rightsquigarrow$ $\pi_{X|E}$ is a homeom.

$$\begin{tikzcd} (x,y) \in K \arrow[rrr, mapsto] & & & y \arrow[ddd, mapsto] \\ X \times Y \arrow[rr, "\pi_Y"] & & Y \arrow[d, two heads, "g"] & \\ X \arrow[u, "(\pi_{X|E})^{-1}"] \arrow[rr, "f"'] \arrow[urr, phantom, "\circlearrowright"] & & Z & \\ x \in X \arrow[uuu, mapsto] \arrow[rrr, mapsto] & & & f(x) = g(y) \end{tikzcd}$$

$\bar{f} = \pi_Y \circ (\pi_{X|E})^{-1}$

"(ii) ⇒ (iii)" $\overset{\text{Lem A}}{\rightsquigarrow}$ $\exists D$ discrete with $g: \beta D \twoheadrightarrow X$ continuous

$$\begin{tikzcd} & \beta D \arrow[d, two heads, "g"] \\ X \arrow[ur, "\bar f"] \arrow[r, "\mathrm{id}"'] \arrow[ur, phantom, "\circlearrowright", shift right=4ex] & X \end{tikzcd}$$

$\rightsquigarrow X \cong \bar f(X) \subseteq \beta D$

$\bar f(X)$ is a retract of βD via $r = \bar f \circ g$

"(iii) ⇒ (i)" D is discrete $\rightsquigarrow$ e.d.
$\overset{\text{FACT 1:}}{\rightsquigarrow}$ βD is e.d. $\overset{\text{FACT 2:}}{\rightsquigarrow}$ X is e.d. $\square$

<u>Thm</u> Every cpt space X has a unique (up to homeom.) projective resolution, i.e. $\exists!$ projective cpt P and $g: P \twoheadrightarrow X$ continuous, s.t. $\forall P' \subsetneqq P$ closed: $g(P') \neq X$.

proof: <u>"$\exists$"</u>: $\overset{\text{Lem A}}{\rightsquigarrow}$ $f: \overset{\text{proj}}{\beta D} \twoheadrightarrow X$ continuous

$\overset{\text{Lem C}}{\rightsquigarrow}$ $\exists\ P \subseteq \beta D$ closed, s.t. $f(P) = X$ and $\forall P' \subsetneqq P$ closed: $f(P') \neq X$. Let $g = f|_P : P \twoheadrightarrow X$, $i: P \hookrightarrow \beta D$

$$\begin{tikzcd} & P \arrow[d, two heads, "g"] \arrow[dl, hook', bend left, "i"] \\ \beta D \arrow[ur, dashed, bend left, "\bar f"] \arrow[r, "f"'] & X \end{tikzcd}$$

$g \circ \bar f = f$ and

$g \circ \underbrace{\bar f \circ i}_{=:j} = f \circ i = g$

Have to show: P proj $\overset{\text{Thm}}{\iff}$ P retract of βD $\Leftarrow$ j is the identity

A: j is not the id.

$\exists p \in P$: $j(p) \neq p$. Take $\underset{p}{U}, \underset{j(p)}{V} \subseteq P$ open, $U \cap V = \emptyset$.

Let $S = P - (\underbrace{U \cap j^{-1}(V)}_{p \in}) \subsetneq P$ closed

and $S \cup j^{-1}(S) = P$

$\leadsto X = g(P) = g(S) \cup g(j^{-1}(S))$

$g(j^{-1}(S)) = g(j(j^{-1}(S))) \subseteq g(S) = X$ ↯ (*)

<u>"uniqueness"</u> $\hat{P}$ proj, $f : \hat{P} \twoheadrightarrow X$ continuous, s.t.

$\forall \hat{P}' \subsetneq P$ closed: $f(\hat{P}') \neq X$.

$$\begin{tikzcd} & \hat{P} \arrow[dl, bend left, "\bar{f}"] \arrow[d, two heads, "f"] \\ P \arrow[ur, bend left, "\bar{g}"] \arrow[r, two heads, "g"'] & X \end{tikzcd}$$

Like above:

$\bar{g} \circ \bar{f} = id_{\hat{P}}$

$\bar{f} \circ \bar{g} = id_P$

$\leadsto P \cong \hat{P}$

$\square$