Compact spaces I

Let X be a Hausdorff topological space.

Recall:
- X is compact if every open cover has a finite subcover
- X is totally disconnected (t.d.) if the only connected non-empty subsets are singletons $\{x\} \subseteq X$
- X is zero-dimensional if every point has arbitrary small closed open neighbourhoods

Ex:
- $\mathbb{Q}$ is t.d.
- Cantor set is t.d. and compact

Prop: Every compact td space is zero-dimensional.

Proof: $x \in X$, V neighbourhood of x, need to show: $\exists\, U \subseteq X$ open, closed: $x \in U \subseteq V$

Assume that V is open $\rightsquigarrow$ $\overline{V}$ is compact

$Q(x) = \bigcap \{\underbrace{D \subseteq \overline{V}}_{=:\mathcal{D}} \mid x \in D,\ D \text{ closed + open in } \overline{V}\}$ $\leftarrow$ quasi-component of x

$\rightsquigarrow$ is connected since $\overline{V}$ is compact $\overset{\overline{V}\text{ td}}{\Longrightarrow}$ $Q(x) = \{x\}$

Let $A = X \setminus V$ $\rightsquigarrow$ A is closed + compact

Further $(X \setminus D)_{D \in \mathcal{D}}$ is an open covering of A $\rightsquigarrow \exists\, D_1, \dots, D_k : A \subseteq \bigcup_{i=1}^{k} X \setminus D_i$

$\Rightarrow \bigcap_{i=1}^{k} D_i \subseteq V$

Let $U := \bigcap_{i=1}^{k} D_i$ closed in X, since closed in $\overline{V}$

open in X, since open in V

$\square$

Prop: Every td compact space embedds into a product of finite spaces
$\rightsquigarrow$ profinite

Proof: Let $\mathcal{D} = \{ D \subseteq X \mid D \text{ closed + open} \}$.

$\forall D \in \mathcal{D}$ let $f_D : X \longrightarrow \mathbb{Z}/2,\ x \longmapsto \begin{cases} 0, & x \in D \\ 1, & x \notin D \end{cases}$ ← continuous since D is open and closed

$\Downarrow$

Define $\Phi : X \longrightarrow \prod_{D \in \mathcal{D}} \mathbb{Z}/2,\ x \longmapsto (f_D(x))_{D \in \mathcal{D}}$ $\rightsquigarrow$ Φ is continuous

Φ is injective:

Let $x, y \in X$, $x \neq y$. Since X is zero-dimensional, there exists $D \subset X$ open and closed such that $x \in D$, $y \notin D$

$\Rightarrow f_D(x) = 0 \neq 1 = f_D(y) \Rightarrow \Phi(x) \neq \Phi(y)$ □

Stone Duality

Def: A compact t.d. space is called a Stone space.

Idea:

category of Boolean Rings → $\mathbf{Bool} \underset{R}{\overset{S}{\rightleftarrows}} \mathbf{Stone}$ ← category of Stone spaces

ring A: $a^2 = a\ \forall a \in A$

$B \longmapsto \mathrm{Hom}_{\mathrm{Ring}}(B, \mathbb{Z}/2)$ ← Ring-homom. $B \longrightarrow \mathbb{Z}/2$ with topology of pointwise convergence $\rightsquigarrow$ Stone space

$C(X, \mathbb{Z}/2) \longleftarrow\!\shortmid X$

continuous functions $X \longrightarrow \mathbb{Z}/2$
$\rightsquigarrow$ Boolean Ring

Thm (Stone, 1937)

The maps $\alpha_X : X \longrightarrow S(R(X)),\ x \longmapsto [f \mapsto f(x)]$, $X \in$ Stone

and $\beta_B : B \longrightarrow R(S(B)),\ b \longmapsto [h \longmapsto h(b)]$, $B \in$ Bool

are isomorphisms in Stone and Bool, resp.

Extremally disconnected spaces

Def: X is extremally disconnected (e.d.), if for every open set $U \subseteq X$ the closure $\overline{U}$ is open in X.

Ex: • $\mathbb{Q}$ is not e.d.
• every discrete space is e.d.
• more next week

Prop: X is e.d. $\Longleftrightarrow$ for every pair $U, V \subseteq X$ open and disjoint, the closures $\overline{U}$ and $\overline{V}$ are disjoint

Proof: „$\Rightarrow$" $U \cap V = \emptyset \Rightarrow U \subset \underbrace{X - V}_{\text{closed}} \Rightarrow \overline{U} \subset X - V \Rightarrow \overline{U} \cap V = \emptyset$

$\Rightarrow V \subset \underbrace{X - \overline{U}}_{\text{closed}} \Rightarrow \overline{V} \subset X - \overline{U} \Rightarrow \overline{V} \cap \overline{U} = \emptyset$

„$\Leftarrow$" Let $U \subseteq X$ be open.

$U \cap (X - \overline{U}) = \emptyset \Rightarrow \overline{U} \cap \overline{(X - \overline{U})} = \emptyset$ — interior of $\overline{U}$

$\Rightarrow \overline{U} \subset X - \overline{(X - \overline{U})} = \operatorname{Int}(\overline{U})$

$\Rightarrow \overline{U} \subseteq \operatorname{Int}(\overline{U}) \Rightarrow \overline{U}$ open $\square$

Prop: Every e.d. space is t.d.

Proof: Let $C \subseteq X$, $C \neq \emptyset$ be a connected subspace. Let $x, y \in C$.

Assume $x \neq y$

$\rightsquigarrow \exists\, U, V \subseteq X$ open : $x \in U$, $y \in V$, $U \cap V = \emptyset \overset{X \text{ e.d.}}{\Rightarrow} \overline{U} \cap \overline{V} = \emptyset$

$\Rightarrow y \notin \overline{U}$

$\overline{U}$ open and closed, hence $\overline{U} \cap C$ is open and closed in C

$\overset{C \text{ connected}}{\Rightarrow}$ $\overline{U} \cap C = C$ (since $x \in \overline{U} \cap C \Rightarrow \overline{U} \cap C \neq \emptyset$)

But $y \in C$ and $y \notin \overline{U}$ ↯ $\Rightarrow x = y$ and $C = \{x\}$ $\Rightarrow X$ t.d.

□

Thm (Halmos 1974)

Let X be a compact t.d. space. Then the dual of X is complete (i.e. every set has infimum and supremum) if and only if X is extremally disconnected