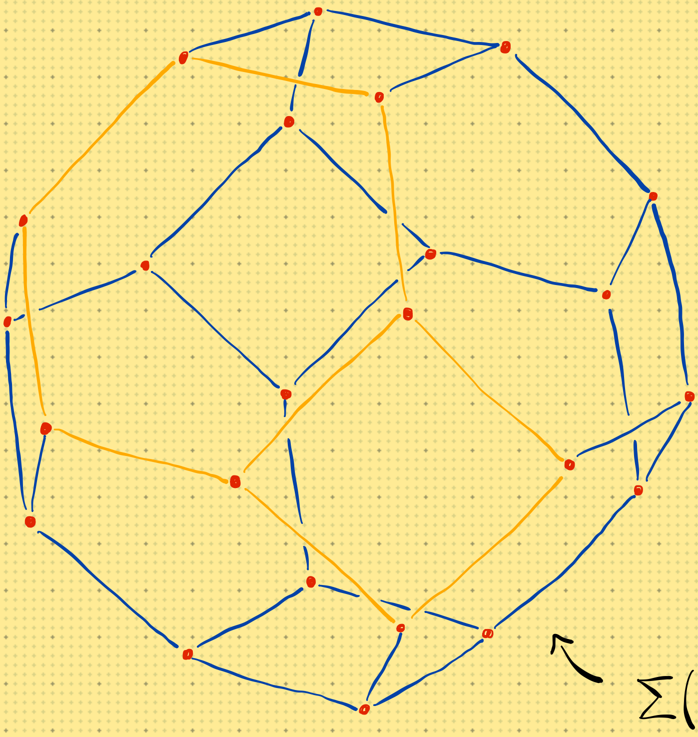
The Davis-Moussong complex Σ

construction, W-action, properties, examples



$\Sigma(S_4, \{\tau_{12}, \tau_{23}, \tau_{34}\})$

Throughout (W, S) is a Coxeter system.

Overview

- Σ is a simplicial complex
- $\Sigma \cong \mathcal{U}(W, K)$
- there is a proper, cocompact W-action on Σ

There is a natural cellulation on Σ such that

- each cell is a Coxeter polytope
- the vertex set is W

- the 1-skeleton is $\mathrm{Cay}(W,S)$
- the 2-skeleton is the Cayley 2-complex of (W,S)
- the link of each vertex is isomorphic to a simplicial complex $L(W,S)$

DEF.:
- a poset is a partially ordered set
- given a poset $\mathcal{P}$ and an element $p \in \mathcal{P}$, put

$$\mathcal{P}_{\leq p} := \{ x \in \mathcal{P} \mid x \leq p \}$$

again a poset

- the flag complex $\mathrm{Flag}(\mathcal{P})$ is defined as follows:

  * the vertex set of $\mathrm{Flag}(\mathcal{P})$ is $\mathcal{P}$
  * $T \subseteq \mathcal{P}$ is a simplex of $\mathrm{Flag}(\mathcal{P})$ $:\Longleftrightarrow$ T is a finite chain in $\mathcal{P}$

$\rightsquigarrow$ $\mathrm{Flag}(\mathcal{P})$ is an abstract simplicial complex

- the geometric realization of $\mathcal{P}$, denoted $|\mathcal{P}|$, is the geometric realization of $\mathrm{Flag}(\mathcal{P})$

- for every $p \in \mathcal{P}$ the subcomplex $|\mathcal{P}_{\leq p}|$ of $|\mathcal{P}|$ is called a face of $|\mathcal{P}|$

DEF.: $T \subseteq S$ is spherical if W_T is a finite subgroup, in this case we say W_T is spherical.

Define

$$\mathcal{S} := \{ T \subseteq S \mid T \text{ is spherical} \}$$

$$W\mathcal{S} := \bigcup_{T \in \mathcal{S}} W/W_T$$

$$= \{ wW_T \mid w \in W,\ T \subseteq \mathcal{S} \}$$

$\rightsquigarrow$ $\mathcal{S}$ and $W\mathcal{S}$ are posets by inclusion

* we have $w'W_{T'} \leq wW_T$ iff $T' \subseteq T$ and $w' \in wW_T$

(Thm. 4.1.6.)

Now

$$|\mathcal{S}| =: K(W,S) \qquad (= K)$$

$$|W\mathcal{S}| =: \Sigma(W,S) \qquad (= \Sigma)$$

Davis-Moussong complex

The inclusion $\mathcal{S} \longrightarrow W\mathcal{S}$, $T \longmapsto W_T$ and
projection $W\mathcal{S} \longrightarrow \mathcal{S}$, $wW_T \longmapsto T$

induce a simplical inclusion $K \longrightarrow \Sigma$ and
simplicial projection $\Sigma \longrightarrow K$.

W acts via leftmultiplication on $W\mathcal{S}$ which induces a W-action on Σ. Given a simplex C in Σ then wC is also a simplex (of same dimension), i.e. the W-action on Σ translates simplices.

EXP.: $W = D_2 = \langle s, t \mid s^2, t^2, (st)^2 \rangle = \{1, s, t, st = ts\}$

sperical subsets: $\emptyset, \{s\}, \{t\}, \{s,t\}$

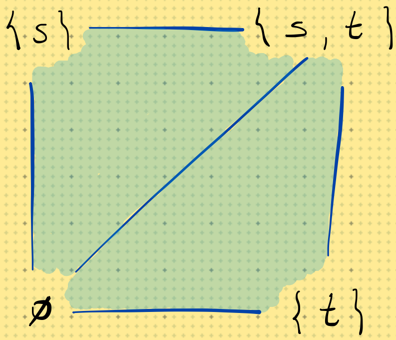
sperical subgroups: $W_\emptyset = \{1\}$ $\quad W_{\{s\}} = \{1, s\}$
$W_{\{t\}} = \{1, t\}$ $\quad W_{\{s,t\}} = W$

Therefore

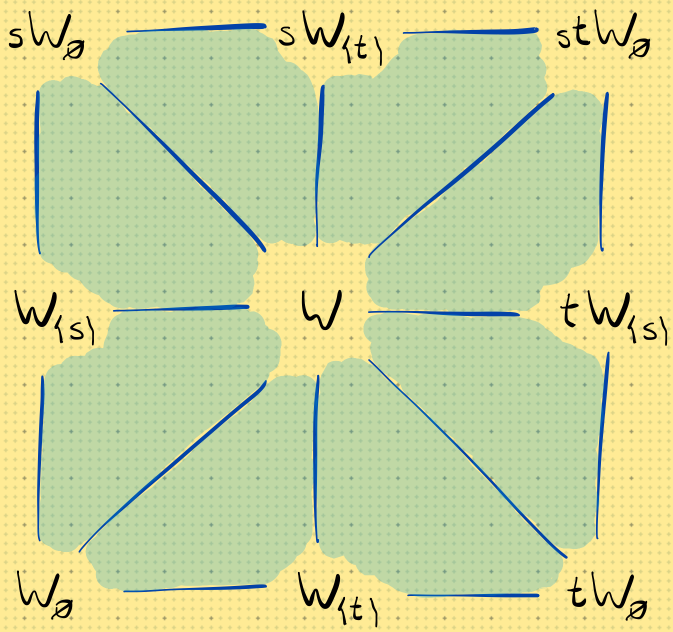
$$\mathcal{S} = \{\emptyset, \{s\}, \{t\}, \{s,t\}\}$$

$$\begin{aligned} W\mathcal{S} = \{ & W_\emptyset, sW_\emptyset, tW_\emptyset, stW_\emptyset, \\ & W_{\{s\}}, tW_{\{s\}}, \\ & W_{\{t\}}, sW_{\{t\}}, \\ & W \} \end{aligned}$$

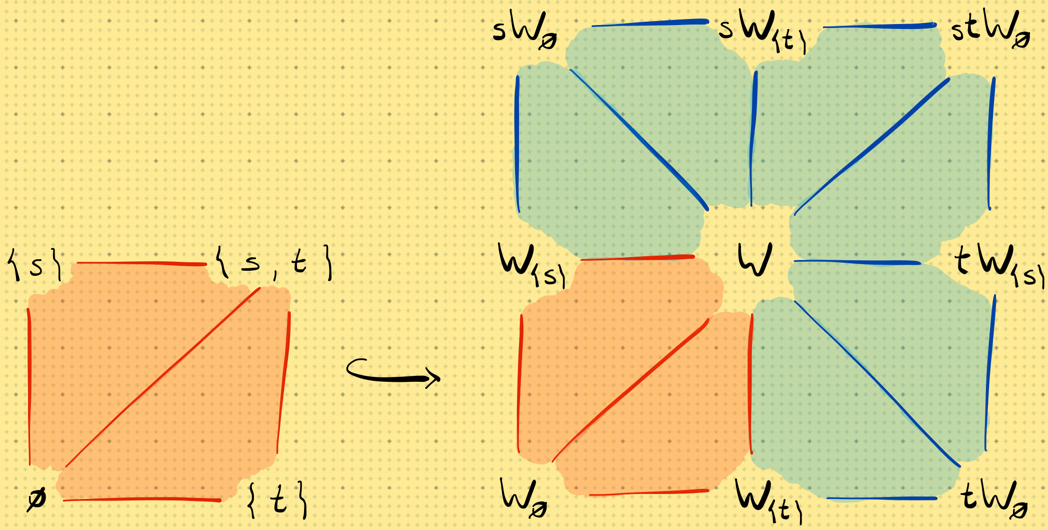
What does $K = |S|$ look like?



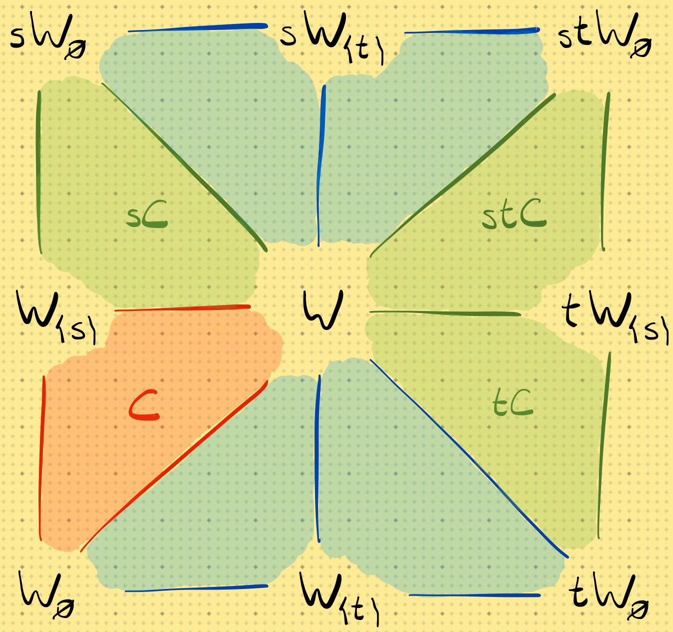
What does $\Sigma = |WS|$ look like?



Recall the above inclusion:



Now the W-action translates simplices and we observe that every simplex in Σ is a translate of a simplex in K. For example:



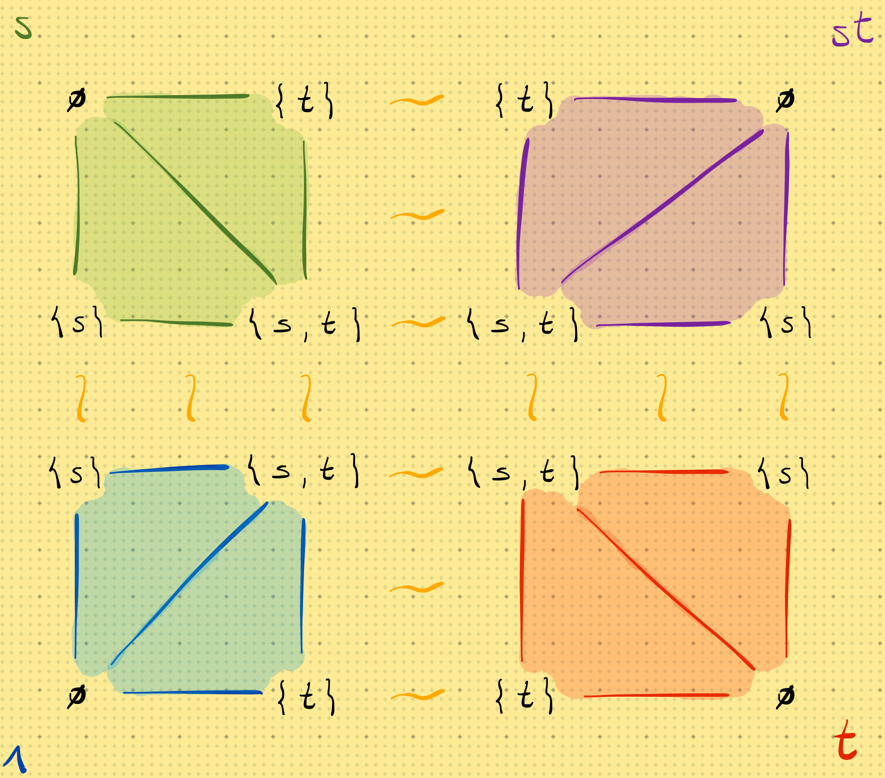
Recall that a mirror structure on a topological space X is a family $\{X_s\}_{s\in S}$ of closed subspaces and that $\mathcal{U}(W,X)$ is defined to be $\mathcal{U}(W,X) := W\times X/_\sim$ with $(h,x)\sim(g,y)$ iff $x=y$ and $h^{-1}g\in W_{S(x)}$ where $S(x)=\{s\in S \mid x\in X_s\}$.

Let's consider the mirror structure $K_s := |\mathcal{S}_{\geq\{s\}}|$ on $K\ (=|\mathcal{S}|)$ in our example:

K_s is the edge $\{s\} \text{———} \{s, t\}$ and

K_t is the edge
$$\begin{tikzcd} \{s,t\} \arrow[d, no head] \\ \{t\} \end{tikzcd}$$
in K.

Now $\mathcal{U}(W, K) := W \times K /\sim$ amounts to one copy of K for every element of W, glued together at these edges:



That is $\Sigma \cong \mathcal{U}(W, K)$.

PROP: Every simplex in Σ is a translate of a simplex in K and K is a fundamental domain of the W-action on Σ.

PROOF: Let $w_0 W_{T_0} \subseteq \dots \subseteq w_k W_{T_k}$ be any simplex in Σ.
Recall $wW_T \subseteq w' W_{T'}$ iff $T \subseteq T'$ and $wW_{T'} = w' W_{T'}$.
So $T_0 \subseteq \dots \subseteq T_k$ and $w_0 W_{T_i} = w_i W_{T_i}$ for all $1 \leq i \leq k$.
Therefore $w_0 W_{T_0} \subseteq \dots \subseteq w_k W_{T_k}$ is the simplex $W_{T_0} \subseteq \dots \subseteq W_{T_k}$ in K translated by w_0.

Let $x \in \Sigma$ and suppose $\{y, z\} \subseteq Wx \cap K$ with $y \neq z$, i.e. $y = gx$ and $z = hx$ with $g \neq h$.
x is contained in a simplex C of maximal dimension in Σ. Then gC and hC are simplices of maximal dimension in K.
Since $y \neq z$ we have $gC \neq hC$.
Let gC (resp. hC) be the simplex $W_{T_0} \subseteq \_ \subseteq W_{T_k}$ (resp. $W_{T_0'} \subseteq \_ \subseteq W_{T_k'}$).
Since K is a cone with cone point $\emptyset$ and the simplices are maximal we have $T_0 = T_0' = \emptyset$.
Now $g^{-1} gC = h^{-1} hC \Rightarrow g^{-1} W_{T_0} = h^{-1} W_{T_0'}$ and so $g^{-1} = h^{-1}$ $\lightning$ $\square$

PROP: There is a W-equivariant homeomorphism $\mathcal{U}(W,K) \to \Sigma$.

PROOF: We identify K as a subspace of Σ via the inclusion $\iota : K \hookrightarrow \Sigma$.

Since $K_s := |\mathcal{S}_{\geq \{s\}}|$ is a union of (closed) simplices in K, it is a closed subspace and $\{K_s\}_{s\in S}$ is a mirror structure on K.

Any simplex in K_s is fixed by s:
if $W_{T_0} \subseteq \dots \subseteq W_{T_k}$ is such a simplex then $W_{\{s\}} \subseteq W_{T_i}$ for all $0 \leq i \leq k$. Therefore $s \in W_{T_i}$ and consequently $sW_{T_i} = W_{T_i}$ for all $0 \leq i \leq k$.
The universal property of the construction $\mathcal{U}$ now extents $\iota : K \hookrightarrow \Sigma$ to a W-equivariant map $\tilde{\iota} : \mathcal{U}(W,K) \to \Sigma$, $[w,x] \mapsto w\,\iota(x)$

Given a W-space Y denote Y^s the fixed point set of $s \in S$ in Y.
Given a map $f : X \to Y$ with $f(X_s) \subseteq Y^s$ there is a unique W-equivariant extension $\bar{f} : \mathcal{U}(W,X) \to Y$ with $\bar{f}([w,x]) = wf(x)$:

$$\begin{tikzcd}
x \in \arrow[d, mapsto] & X \arrow[r, "f"] \arrow[d, hook] & Y \\
{[1,x]} \in & \mathcal{U}(W,X) \arrow[ur, dashed, "\exists! \bar{f}"'] &
\end{tikzcd}$$

With the above proposition $\tilde{\iota}$ is bijective and since K has the subspace topology of Σ and $\mathcal{U}(W, K)$ is just copies of K glued together the topologies are equal and $\tilde{\iota}$ is a homeomorphism. $\square$

PROP: Considering W with the discrete topology, the W-action on Σ is proper and cocompact if S is finite.

PROOF: Since the W-action on Σ is translation of simplices, every $w \in W$ induces a homeomorphism $\Sigma \longrightarrow \Sigma$; since W has discrete topology the action is continuous.

Properness:

Using Lemma 5.1.7. from Davis' book and above proposition we only have to show that K is Hausdorff and the mirror structure $\{K_s\}_{s\in S}$ is W-finite, i.e. $\bigcap_{s\in T} K_s = \emptyset$ whenever $T \subseteq S$ is not spherical.

1. K is Hausdorff since we can identify it with the geometric realization of $\mathcal{S}$ in $\mathbb{R}^S$.
2. if $T \subseteq S$ is not spherical, i.e. $T \notin \mathcal{S}$, then $\bigcap_{s\in T} K_s = \bigcap_{s\in T} |\mathcal{S}_{\geq\{s\}}| = |\bigcap_{s\in T} \mathcal{S}_{\geq\{s\}}| = |\mathcal{S}_{\geq T}| = \emptyset$

   since $\mathcal{S}_{\geq T} = \emptyset$.

Cocompactness:

We have to show that the orbit space of the W-action on Σ is compact.

Using the above proposition and the fact, that the orbit space of the W-action on $\mathcal{U}(W,K)$ is homeomorphic to K, we only have to show that K is compact.

This is the case, since K is a finite simplicial complex since S is finite.

$\square$

From the definition Σ is a simplicial complex, but there is a coarser cell structure given by the faces $|\mathcal{WS}_{\le wW_T}|$:

PROP.: i) $|\mathcal{WS}_{\le wW_T}|$ is a union of simplices of Σ

ii) The intersection of two faces is either empty or a common face of the intersected faces

iii) Σ is the union of all faces

PROOF: i) $\mathcal{WS}_{\le wW_T}$ is the union of all finite chains in $\mathcal{WS}$ with maximal element wW_T.
Since finite chains in $\mathcal{WS}_{\le wW_T}$ resp. $\mathcal{WS}$ give the simplices in $|\mathcal{WS}_{\le wW_T}|$ resp. Σ this assertion holds.

ii) Follows from the following lemma and the fact

$$|\mathcal{WS}_{\le w'W_{T'}}| \cap |\mathcal{WS}_{\le w''W_{T''}}|$$
$$= |\mathcal{WS}_{\le w'W_{T'}} \cap \mathcal{WS}_{\le w''W_{T''}}|$$

iii) Trivial since $W \in \mathcal{WS}$ and $\mathcal{WS}_{\le W} = \mathcal{WS}$. $\square$

LEMMA: We have either

$$\mathcal{WS}_{\leq w'W_{T'}} \cap \mathcal{WS}_{\leq w''W_{T''}} = \emptyset$$

or

$$\mathcal{WS}_{\leq w'W_{T'}} \cap \mathcal{WS}_{\leq w''W_{T''}} = \mathcal{WS}_{\leq wW_T}$$

for some $wW_T \in \mathcal{WS}$.

PROOF: We have

$$\mathcal{WS}_{\leq w'W_{T'}} \cap \mathcal{WS}_{\leq w''W_{T''}}$$

$$= \{\tilde{w}W_{\tilde{T}} \mid \tilde{w}W_{\tilde{T}} \leq w'W_{T'} \text{ and } \tilde{w}W_{\tilde{T}} \leq w''W_{T''}\}$$

$$= \{\tilde{w}W_{\tilde{T}} \mid \tilde{T} \subseteq T' \cap T'' \text{ and } \tilde{w} \in w'W_{T'} \cap w''W_{T''}\}.$$

Suppose the intersection is not empty, then we have $w \in w'W_{T'} \cap w''W_{T''}$ for some $w \in W$ and with the following lemma we have

$$w'W_{T'} \cap w''W_{T''} = w(W_{T'} \cap W_{T''})$$
$$= wW_{T' \cap T''}$$

As a result we have

$$\mathcal{WS}_{\leq w'W_{T'}} \cap \mathcal{WS}_{\leq w''W_{T''}}$$
$$= \{\tilde{w}W_{\tilde{T}} \mid \tilde{T} \subseteq T' \cap T'' \text{ and } \tilde{w} \in wW_{T' \cap T''}\}$$
$$= \mathcal{WS}_{\leq wW_{T' \cap T''}}.$$

$\square$

LEMMA: Let G be a group and H and K subgroups. The intersection of a coset of H and a coset of K is either empty or a coset of $H \cap K$.

PROOF: Suppose $x \in aH \cap bK$. Since $x \in aH$ and $x \in bK$ we have $xH = aH$ and $xK = bK$.
Then $aH \cap bK = xH \cap xK = x(H \cap K)$ with the last identy holding since

$$y \in xH \cap xK \iff \exists\, d \in H \cap K \text{ with } y = xd$$
$$\iff y \in x(H \cap K)$$

$\square$

So each cell of this coarser cell structure is given by a face $|W\mathcal{S}_{\leq wW_T}|$ and therefore corresponds to a coset wW_T.

What does $|W\mathcal{S}_{\leq wW_T}|$ look like?

PROP: We have $|W\mathcal{S}_{\leq wW_T}| \cong \Sigma(W_T, T)$.

PROOF: We have

$$W_T\mathcal{S}_{\leq T} = \bigcup_{T' \in \mathcal{S}_{\leq T}} W_T/W_{T'}$$

$$= \{w'W_{T'} \mid w' \in W_T \text{ and } T' \leq T\}.$$

Using the bijection $wW_T \to W_T$, $wv \mapsto v$ we get an isomorphism of posets

$W\mathcal{S}_{\leq wW_T} \longrightarrow W_T\mathcal{S}_{\leq T}$, $w'W_{T'} \mapsto vW_{T'}$

with $w' = wv$ with $v \in W_T$ since $w' \in wW_T$.

As a result

$$|W\mathcal{S}_{\leq wW_T}| \cong |W_T\mathcal{S}_{\leq T}| = \Sigma(W_T, T).$$

$\square$

What does Σ look like for finite W?

EXP.: $W = D_3 \cong S_3 \cong \langle s,t \mid s^2, t^2, (st)^3 \rangle$

spherical subgroups of W:

$$W_\emptyset = \langle \emptyset \rangle = \{1\}, \quad W_{\{s\}} = \langle s \rangle = \{1, s\}$$

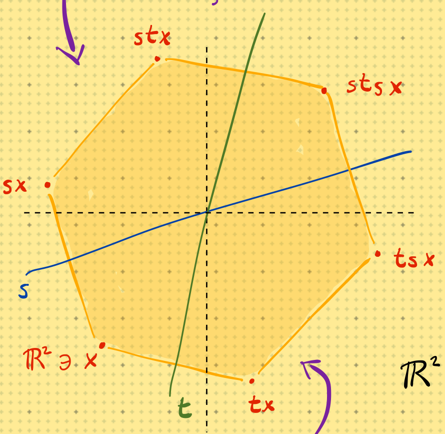
$$W_{\{t\}} = \langle t \rangle = \{1, t\}, \quad W_{\{s,t\}} = \langle s,t \rangle = W$$

spherical cosets:

$$\begin{aligned} W\mathcal{S} = \{ & W_\emptyset, sW_\emptyset, tW_\emptyset, stW_\emptyset, tsW_\emptyset, stsW_\emptyset, \\ & W_{\{s\}}, tW_{\{s\}}, stW_{\{s\}}, \\ & W_{\{t\}}, sW_{\{t\}}, tsW_{\{t\}}, \\ & W_{\{s,t\}} \} \end{aligned}$$

$$\begin{aligned} = \{ & \{1\}, \{s\}, \{t\}, \{st\}, \{ts\}, \{sts\}, \\ & \{1, s\}, \{t, ts\}, \{st, sts\}, \\ & \{1, t\}, \{s, st\}, \{ts, tst\}, \\ & \{1, s, t, st, ts, sts\} \} \end{aligned}$$

canonical representation of D_3 on $\mathbb{R}^2$



Coxeter polytope / Coxeter cell

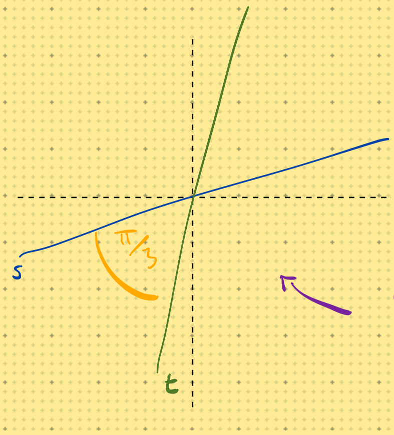
$\rightsquigarrow$ each coset in $W\mathcal{S}$ corresponds to a face of the polygon on the right

DEF./PROP.: For every Coxeter system (W, S) there is a linear representation $W \longrightarrow \mathrm{Aut}(\mathbb{R}^S)$ that maps each generating involution of W to a reflection across a hyperplane in $\mathbb{R}^S$.

This representation is called the canonical representation of (W, S).

sketch of construction:

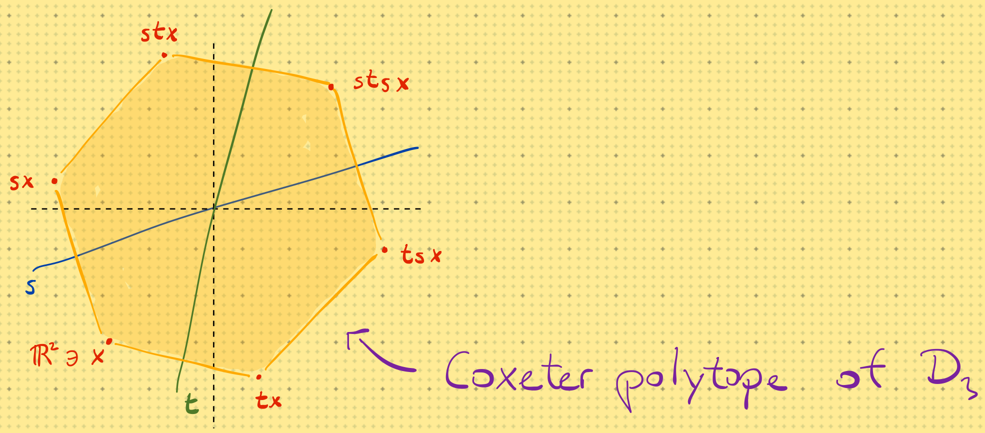
- the Coxeter matrix M defines a symmetric bilinear form $B_M(e_i, e_j) := -\cos(\pi / m_{ij})$ on $\mathbb{R}^S$
- for each $s \in S$ define a hyperplane $H_s := \{ x \in \mathbb{R}^S \mid B_M(x, e_s) = 0 \}$ and a reflection ρ_s across H_s
- the mapping $s \mapsto \rho_s$ extends to a homomorphism $W \longrightarrow \mathrm{Aut}(\mathbb{R}^S)$



canonical representation of D_3 on $\mathbb{R}^2$

DEF: Given a point x in the interior of the fundamental domain of the canonical representation.

The Coxeter polytope / Coxeter cell associated to (W,S) with W finite is the convex polytope defined as the convex hull of the orbit Wx.

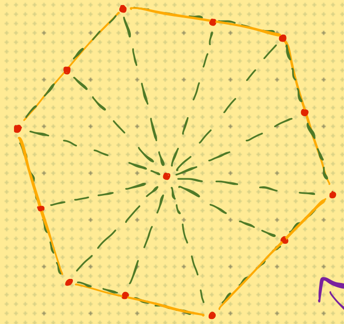


DEF / PROP: The barycentric subdivision of a polytope P is defined as follows:

- for every face $F \in \mathcal{F}(P)$ choose a point v_F in it's interior
- every chain $F_1 \subseteq \dots \subseteq F_k$ in $\mathcal{F}(P)$ defines a simplex spanned by $(v_{F_1}, \dots v_{F_k})$

It is a simplicial complex.

$\rightsquigarrow$ in other words the barycentric subdivision is $|\mathcal{F}(P)|$



barycentric subdivision of a hexagon

Since $\Sigma = |\mathcal{W}\mathcal{S}|$ and $bP = |\mathcal{F}(P)|$, using the next proposition we can identify Σ for finite W as the barycentric subdivision of the associated Coxeter polytope.

PROP: Given a finite Coxeter system (W, S) and it's Coxeter polytope C the mapping

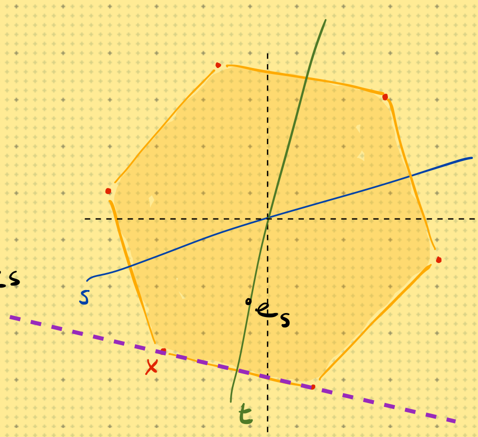
$$\mathcal{W}\mathcal{S} \to \mathcal{F}(C), \; wW_T \mapsto \langle wW_T \cdot x \rangle$$

$$\left(= \langle \{ \tilde{w} x \mid \tilde{w} \in wW_T \} \rangle \right.$$

convex hull of wW_T, i.e. a subset of $\mathbb{R}^S$

is an isomorphism of posets.

PROOF: (Lemma 7.3.3. in Davis' book)

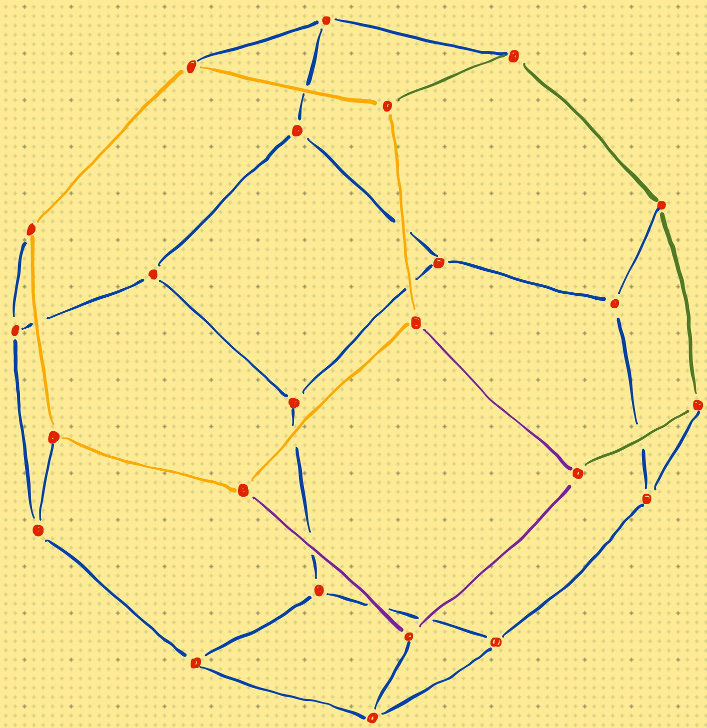


Sketch: Fix $s \in S$.

$\{ v \in \mathbb{R}^2 \mid \langle v - x, e_s \rangle = 0 \}$ defines a supporting hyperplane of C which corresponds to the codimension one face spanned by $W_T x$ with $T = S - \{s\}$.

Varying s and replacing x with wx and e_s with we_s we get a description of all supporting hypesplanes of C.

EXP.: $\Sigma\left(S_4, \{\tau_{12}, \tau_{23}, \tau_{34}\}\right)$



spherical subgroups	associated Coxeter polytope
$\langle \emptyset \rangle \cong \{1\}$	0 - simplex
$\langle \tau_{12} \rangle \cong \langle \tau_{23} \rangle \cong \langle \tau_{34} \rangle \cong C_2$	1 - simplex
$\langle \tau_{12}, \tau_{34} \rangle \cong D_2$	rectangle
$\langle \tau_{12}, \tau_{23} \rangle \cong \langle \tau_{23}, \tau_{34} \rangle \cong S_3 \cong D_3$	hexagon
$\langle \tau_{12}, \tau_{23}, \tau_{34} \rangle = S_4$	permutohedron

Summary:

There is a cell structure on Σ such that

$\boxed{1}$ each cell corresponds to a spherical coset wW_T and is a Coxeter polytope of type W_T with dimension equal to the cardinality of T

$\boxed{2}$ the vertex set is W

Since cosets $wW_\emptyset$ i.e. elements of W correspond to vertices

$\boxed{3}$ the 1-skeleton is $Cay(W, S)$

Since cosets $wW_{\{s\}}$ for $s \in S$ correspond to 1-simplices connecting w and ws

$\boxed{4}$ the 2-skeleton is the Cayley 2-complex of (W, S)

sketch: The Cayley 2-complex is constructed by gluing a 2-disk into each loop of $Cay(W, S)$.

Every loop in $Cay(W, S)$ is given by a relation $(s_i s_j)^{m_{ij}}$ with $i \neq j$ and in Σ

there is a 2-cell of type $W_{\{s_i, s_j\}}$ glued into this loop.

$\rightsquigarrow$ since the Cayley 2-complex is simply connected and $\pi_1(\Sigma)$ only depends on the 2-skeleton, Σ is simply connected

$\boxed{5}$ the link of each vertex is isomorphic to the nerve L of (W, S)

The nerve of (W, S) is defined as the geometric realization of the abstract simplicial complex $\mathcal{S}_{>\emptyset}$.

We can identify the link of a vertex v with the subcomplex C_v of all cells containing v.

Mapping each cell of type W_T to T gives an isomorphism

$$C_v \longrightarrow L, \quad wW_T \longmapsto T.$$

Remarks

- Let (W,S) be a geometric reflection group generated by a polytope P in $\mathbb{E}^n$ or $\mathbb{H}^n$, then

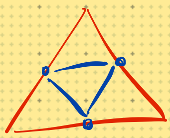
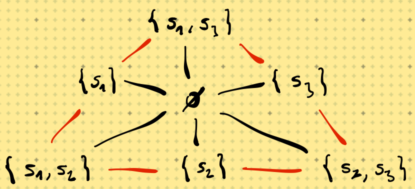
  $\rightsquigarrow$ we can identify the nerve L with the boundary complex of the dual polytope of P

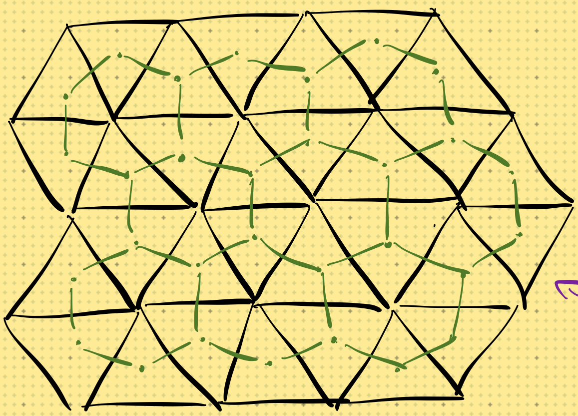
  $\rightsquigarrow$ $K \cong P$ and therefore $\Sigma \cong \mathcal{U}(W,P) \cong \mathbb{E}^n$ or $\Sigma \cong \mathbb{H}^n$ respectively

  $\rightsquigarrow$ the cell structure on Σ is dual to the tessalation of $\mathbb{E}^n$ resp. $\mathbb{H}^n$ by P

Examples: $P = \triangle$ in $\mathbb{E}^2$

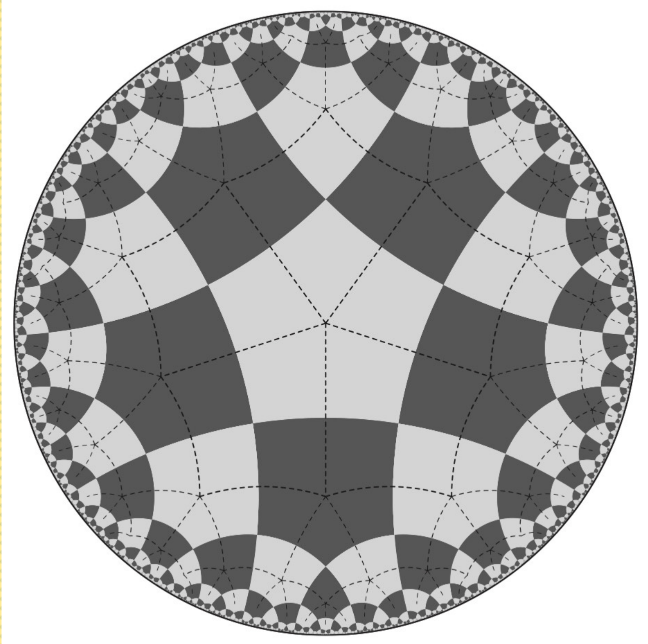
$\rightsquigarrow$ $W \cong \langle s_1, s_2, s_3 \mid s_1^2, s_2^2, s_3^2, (s_1 s_2)^3, (s_1 s_3)^3, (s_2 s_3)^3 \rangle$

$\rightsquigarrow$ $L \cong$  $K =$ 



cell structure on Σ

P right-angled pentagon in $\mathbb{H}^2$



- There are many Coxeter systems

  (!) given a cell complex Λ there is a right-angled Coxeter system with nerve the barycentric subdivision of Λ

  but only very few arise as geometric reflection groups on $\mathbb{E}^n$ or $\mathbb{H}^n$.

  Since Σ is always defined and W acts on it properly and cocompactly as a reflection group with fundamental domain K it can be viewed as a satisfactory replacement for the constant curvature spaces.