7. Übungszettel zur Vorlesung „Räume nichtpositiver Krümmung“

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WWU Münster

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Each question is worth 4 points.

Aufgabe 7.1 (Isometries of trees)
Prove that every isometry g of an $\mathbb{R}-$tree is semi-simple.

Hint: Let $x \in X$ and let m be the midpoint of x and $g(x)$.
Prove that $d(g(m), m) = l_g$.

Aufgabe 7.2 (Translation lengths)
Let (X, d) be a metric space and g be an isometry of X. Consider the limit

$$\lim_{n\to\infty} \frac{d(x, g^n(x))}{n}.$$

a) Show that the limit exists for every $x \in X$ using the following steps.

   i) Show that $f(n) = d(x, g^n(x))$ is a subadditive function, that is, $f(m + n) \leq f(m) + f(n)$ for all m and n.

   ii) Use (i) to show that the limit $\lim_{n\to\infty} \frac{f(n)}{n}$ exists.
   Hint: For $d \in \mathbb{N}$, express $n = qd + r$, where $0 \leq r < d$.

b) Show that the limit is independent of $x \in X$.

c) If X is CAT(0) and g is semi-simple, then show that for any $x \in X$,

$$l_g = \lim_{n\to\infty} \frac{d(x, g^n(x))}{n}.$$

Hint: Use the convexity of μ_g.

Aufgabe 7.3 (Flat strips in $\mathbb{H}^m$)
Show that there are no flat strips in the hyperbolic space $\mathbb{H}^m$, that is, show that there is no isometric embedding $\mathbb{R} \times [0, D] \to \mathbb{H}^m$ for any $D > 0$.

Aufgabe 7.4 (Flat triangles)
Let a, b, c be distinct points in a CAT(0) space X, with comparison points $\bar{a}, \bar{b}, \bar{c}$ in $\mathbb{R}^2$. Suppose that there is a point z on the geodesic from b to c, different from b and c, with $d(z, a) = ||\bar{z} - \bar{a}||_2$.

Show that there is a unique isometric embedding ϕ of the convex hull D of $\bar{a}, \bar{b}, \bar{c}$ in $\mathbb{R}^2$ into X, such that $\phi(\bar{a}) = a$, $\phi(\bar{b}) = b$, and $\phi(\bar{c}) = c$.

7.*-Aufgabe (The Sandwich Lemma)
Let X be a CAT(0) space. For a closed subspace $C \subset X$, write $d_C(x) = \inf\{d(x,c) \mid c \in C\}$ to denote the distance of a point x from C.

Let C_1 and C_2 be two complete, convex subspaces of X. Prove that if the restriction of d_{C_1} to C_2 is constant, equal to say a, and the restriction of d_{C_2} to C_1 is constant, then the convex hull of $C_1 \cup C_2$ is isometric to $C_1 \times [0, a]$.

Abgabe bis: Donnerstag, den 7.1.2021, 8 Uhr online im Learnwebkurs.