6. Übungszettel zur Vorlesung „Räume nichtpositiver Krümmung“

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WWU Münster

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Each question is worth 4 points.

Aufgabe 6.1 (Isometries of the Euclidean space)
Show that every isometry of the Euclidean n-space $\mathbb{R}^n$ is of the form $v \mapsto Av+b$, where b is a vector in $\mathbb{R}^n$ and A is an $n \times n$ orthogonal matrix.

Hint: It suffices to show that an isometry that fixes the origin is a linear map.

Aufgabe 6.2 (Isometries of L^2)
In the Hilbert space $L^2(\mathbb{Z})$, let a be the shift operator given by $a(f)(i) = f(i-1)$, where $f \in L^2(\mathbb{Z})$ and $i \in \mathbb{Z}$. Let $t \in L^2(\mathbb{Z})$ be the vector with the only entry 1 at $i = 0$. Then show that $f \mapsto a(f) + t$ is a parabolic isometry.

Aufgabe 6.3 (Parabolic isometries of $\mathbb{H}^n$)

a) Let $\widehat{\mathbb{R}^m} := \mathbb{R}^m \cup \{\infty\}$.

   Consider the stereographic projection $s : S^{n-1} \to \widehat{\mathbb{R}^{n-1}}$ from the south pole $P = (0, \ldots, 0, -1)$.

   i) Show that s is the restriction of the inversion map $\iota : \widehat{\mathbb{R}^n} \to \widehat{\mathbb{R}^n}$ in the sphere with center P and radius $\sqrt{2}$.
   ii) Find an expression for this inversion map ι.
   iii) Show that the restriction of ι to B^n is a homeomorphism $c : B^n \to H^n$ called the *Cayley transform*, where $B^n = \{x \in \mathbb{R}^n \mid ||x||_2 < 1\}$ is the unit ball in $\mathbb{R}^n$ and $H^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$ is the upper half space.

   Let (B^n, d) be the Poincaré ball model (Exercise 4.4) of the hyperbolic n-space. We put a metric $\tilde{d}$ on H^n so that c becomes an isometry. Then $(H^n, \tilde{d})$ is called the *Poincaré Half space model* of the hyperbolic n-space $\mathbb{H}^n$.

b) Use the half space model to give an example of a parabolic isometry of the hyperbolic space $\mathbb{H}^m$, $m \geq 2$.

Aufgabe 6.4 (Homogeneity of $\mathbb{H}^n$)
Prove that $\mathbb{H}^n$ is 2-point homogeneous, that is, given two pairs of points (x_1, x_2)

and (y_1, y_2) in $\mathbb{H}^m$ such that $d(x_1, x_2) = d(y_1, y_2)$, there is an isometry mapping one pair (x_1, x_2) to the other, (y_1, y_2).

6.*-Aufgabe (Bounded convex functions)
A bounded convex function $f : \mathbb{R} \to \mathbb{R}$ is constant.

Abgabe bis: Donnerstag, den 17.12.2020, 8 Uhr online im Learnwebkurs.