5. Übungszettel zur Vorlesung „Räume nichtpositiver Krümmung“

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Each question is worth 4 points.

Aufgabe 5.1 (Angles)
Recall the definition of angle: Let X be a metric space and let $c : [0, a] \to X$ and $c' : [0, a'] \to X$ be two geodesic paths with $c(0) = c'(0)$. Given $t \in (0, a]$ and $t' \in (0, a']$, consider the comparison triangle $\overline{\Delta}(c(0), c(t), c'(t'))$ in the Euclidean plane, and the comparison angle $\overline{\angle}_{c(0)}(c(t), c'(t'))$. The (Alexandrov) *angle* between c and c' is given by

$$\angle(c, c') = \limsup_{t,t' \to 0} \overline{\angle}_{c(0)}(c(t), c'(t')) = \lim_{\epsilon \to 0} \sup_{0 < t, t' < \epsilon} \overline{\angle}_{c(0)}(c(t), c'(t')).$$

Consider the metric space $(\mathbb{R}^2, || - ||_\infty)$. For every integer $n > 1$, the map $c_n : [0, 1/n] \to (\mathbb{R}^2, || - ||_\infty)$, $t \mapsto (t, [t(1-t)]^n)$ defines a geodesic path. Show that all these geodesics are pairwise disjoint, but the angle between any two of them is zero.

Aufgabe 5.2 (Concatenation of geodesics)
Let X be a CAT(0) space. If $p, x, y \in X$, then the geodesic segment $[x, y]$ is the union of $[x, p]$ and $[p, y]$ if and only if $\angle(x, y) = \pi$.

Aufgabe 5.3 (CAT(0) normed vector spaces)

a) Show that a normed vector space V is an inner product space if and only if the angle is defined in the strict sense, that is, the limit

$$\lim_{t,t' \to 0} \overline{\angle}_0(c(t), c'(t'))$$

exists for all pairs of geodesics c, c' issuing from $0 \in V$.

Hint: Use the parallelogram law.

b) Show that a normed vector space is CAT(0) if and only if it is an inner product space.

Aufgabe 5.4 (Gluing metric spaces)
Let $X = X_1 \cup X_2$ be a set. Suppose that d_i is a metric on X_i, for $i = 1, 2$, such that d_1 and d_2 agree on $A = X_1 \cap X_2$ and that A is nonempty and closed in X_1 and in X_2. We define a pseudometric d on X as follows. We put $d(p, q) = d_i(p, q)$

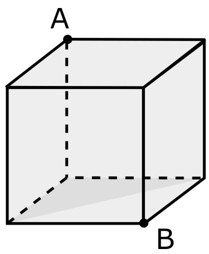
if p, q are in the same set X_i, and $d(p,q) = \inf\{d(p,a) + d(a,q) | a \in A\}$ in the remaining case.

Show that d is a metric on X. Show that X is geodesic if X_1 and X_2 are geodesic and if A is proper. Show that this may fail if A is not assumed to be proper.

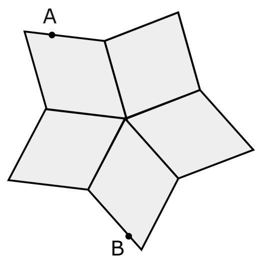
5.*-Aufgabe (Santa Claus' robot)

(A) Let $C_2 = [0,1]^2$ be a unit square, and $C_3 = [0,1]^3$ be a unit cube with the Euclidean metric. In this exercise, we glue copies of such squares and cubes together in order to create new metric spaces.

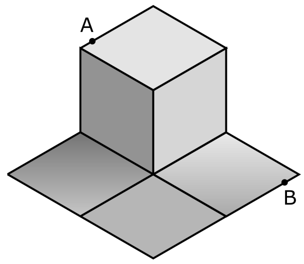
Consider the following spaces X.



(a) six copies of C_2 glued to create the boundary of the 3-cube C_3



(b) five copies of C_2 glued with a common vertex



(c) three copies of C_2 glued to three edges of C_3

For each of the above spaces,

a) Sketch a path of shortest length between the given points A and B.

b) Is there a unique path of minimum length? Explain.

c) Can you find such a unique path of minimum length, for any two points P and Q in the space? Explain.

d) Describe the length of this path between P and Q.

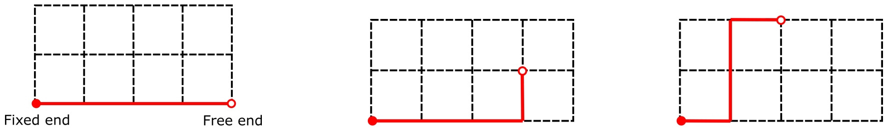
e) This length, in fact, gives a metric on the space X. Which of the above spaces under this metric will become a CAT(0) space? Make a guess.

Similarly, given different metrics on compact spaces, one can create a new metric space by gluing them together along isometric subspaces.

(B) In today's high tech world, Santa Claus also has a robot helper. This robot has an arm of length 4 (with 4 links of unit length) that moves in a "tunnel" of height 2. Below are different positions of the robot arm.

The arm is affixed to the lower left corner of the tunnel and it can move freely without colliding with itself using the following two moves:

- Flipping a corner: Two consecutive links facing different directions interchange directions.



(a) (b) (c)

- Rotating the end: The last link of the arm rotates by 90 degrees without intersecting itself.

We build the "map of possibilities" of the robot as a cubical complex X with a vertex for for each position of the arm, and an edge for each local move between two positions. Suppose for two such moves M and M', performing move M and then move M' results in the same position as performing move M' and then move M. In this case, we get a square in the graph and we fill in this square.

If we wish to move the robot efficiently, we should let it perform various moves simultaneously. In our complex, this corresponds to walking across the diagonal of the corresponding cube.

a) Find all the different positions that the robot arm can assume and draw the cube complex described above.

b) Suppose the time required to make one of the above moves is T. Pick two positions (vertices in the complex) A and B. What is the time required to move from position A to B?

c) Can you generalize this to develop a metric on the cubical complex X that will measure the time required to move from position A to position B?

Abgabe bis: Donnerstag, den 10.12.2020, 8 Uhr online im Learnwebkurs.