2. Übungszettel zur Vorlesung „Räume nichtpositiver Krümmung“

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Each question is worth 4 points.

Aufgabe 2.1
Let $X = \mathbb{R}$ with the usual metric.

a) Determine the mutual Hausdorff distances between the subgroups $\mathbb{R}$, and the subgroups $a\mathbb{Z}$, for all $a \geq 0$.

b) Is the set $\{\mathbb{R}\} \cup \{a\mathbb{Z} \mid a \geq 0\}$ closed with respect to the Hausdorff distance?

Hint: For $0 < a < b$, consider the image of $a\mathbb{Z}$ in $\mathbb{R}/b\mathbb{Z}$.

Aufgabe 2.2
A metric space X *has midpoints* if for all points $p, q \in X$ there is m in X with $d(p, m) = d(m, q) = d(p, q)/2$. Show that a complete metric space with midpoints is geodesic.

Hint: For $p, q \in X$ with $d(p, q) = r$ construct first isometric embeddings of $J_0 = \{0, r\}, J_1 = \{0, r/2, r\}, J_2 = \{0, r/4, 2r/4, 3r/4, r\}$ etc. Then use Satz 1.5 to extend to $[0, r]$.

Aufgabe 2.3 (Normed spaces)

a) Show that every normed vector space is a geodesic space. Further, it is uniquely geodesic if and only if the unit ball is strictly convex, that is, if v_1 and v_2 are distinct vectors of norm 1, then $||(1-t)v_1 + tv_2|| < 1$ for all $t \in (0, 1)$.

b) Consider $\mathbb{R}^n$ with the norms $|| \cdot ||_1$, $|| \cdot ||_2$, and $|| \cdot ||_\infty$. Which of these normed spaces are uniquely geodesic?

c) Describe the geodesics in $(\mathbb{R}^2, || \cdot ||_1)$ and $(\mathbb{R}^2, || \cdot ||_\infty)$.

Aufgabe 2.4
Let $X = \mathbb{R}^2$ with the following metric d,

$$d((x_1, y_1), (x_2, y_2)) = \begin{cases} |y_1 - y_2| & \text{if } x_1 = x_2 \\ |y_1| + |x_1 - x_2| + |y_2| & \text{else} \end{cases}$$

Show that X is a CAT(0) space.

Hint: First determine what the geodesics in X look like.

***-Aufgabe**
Let A be a convex subset of a CAT(0) space. Show that the closure $\bar{A}$ is also convex.

Hint: For $s \in [0, 1]$, let $f : X \times X \to X$ denote the map that sends (p, q) to the unique point u on the geodesic from p to q with $d(p, u) = sd(p, q)$. Show that f is continuous.

Abgabe bis: Donnerstag, den 19.11.2020, 8 Uhr online im Learnwebkurs