Some results on geometric reflection groups – Lannér's and Andreev's Theorem

Reminder: geometric reflection groups

A geometric reflection group is a group W with:

- $W \curvearrowright \mathbb{X}^n$ ($\Rightarrow$ proper)
- W is generated by a convex, simple polytope P

$\rightsquigarrow$ W has (strict) fundamental domain P

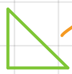
The solutions of

$$\frac{\pi}{m_1} + \frac{\pi}{m_2} + \dots + \frac{\pi}{m_k} \overset{?}{\substack{> \\ = \\ <}} \pi$$

gave us (polygon) reflection groups in $\mathbb{X}^2$

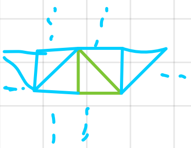
Euclidean case: $(2,3,6)$ 

$(3,3,3)$

$(2,4,4)$  simplex

Question: $P^n = \Delta^n$ $\quad n \geq 3$?

Definition: simplicial Coxeter group



A Coxeter group W is simplicial if $W \curvearrowright \mathcal{U}(W, \Delta^n)$ proper with fundamental domain a simplex

Lannér's Theorem

1.) Any simplicial Coxeter group can be represented as geometric reflection group with fundamental chamber an n-simplex in $\mathbb{X}^n$.

2.) The spherical and Euclidean Coxeter diagram's can be completly classified.

└▷ next week.

Definition: Gram matrix

The Gram matrix of a $\begin{pmatrix}\text{spherical}\\ \text{Euclidean}\\ \text{hyperbolic}\end{pmatrix}$ simplex σ with unit inward-pointing normals $u_0, \dots, u_n$ is a matrix $C(\sigma) \in \mathbb{R}^{(n+1)\times(n+1)}$ s.t.

$$C_{ij}(\sigma) := \langle u_i, u_j \rangle \qquad , i,j = 0, \dots, n+1 :$$

Question:

Can we find a $\begin{matrix}\text{spherical}\\ \text{Euclidean}\\ \text{hyperbolic}\end{matrix}$ simplex for any given dihedral angles $\theta_{ij} \in (0, \pi)$ along $\sigma_i \cap \sigma_j$?

If such a simplex exists its Gram matrix would be the matrix $C(\theta)$ defined by

$$C_{ij}(\theta) = \begin{cases} 1 & , i = j \\ -\cos\theta_{ij} & , i \neq j \end{cases}$$

Lemma 5

σ is a spherical simplex with dihedral angles predescribed by (θ_{ij}) if and only if its Gram matrix $C(\theta)$ is positive definite.

proof: "$\Rightarrow$" Note that the Gram matrix C of a spherical simplex σ can be rewritten as

$$C = U^T U \quad , \quad \text{where } U = (\underbrace{u_0 \,|\, \dots \,|\, u_n}_{\text{unit inward-pointing normals.}})$$

Since $\{u_0, \dots, u_n\}$ is a basis of $\mathbb{R}^{n+1}$, U is nonsingular. Thus. C is positive definite. #

"$\Leftarrow$"

Lemma

Every spherical simplex is uniquely determined by its Gram matrix, up to isometry.

proof:

Let σ, σ' be spherical n-simplices with the same Gram matrix, i.e.

$$\underline{\langle u_i, u_j \rangle = \langle u_i', u_j' \rangle} \quad \text{for all } i,j = 0, \dots, n$$

Since $\{u_0, \dots, u_n\}$ and $\{u_0', \dots, u_n'\}$ are two bases of $\mathbb{R}^{n+1}$, there ex. a unique linear automorphism

$$g : \mathbb{R}^{n+1} \xrightarrow{\sim} \mathbb{R}^{n+1} \quad \text{with} \quad \underline{g(u_i) = u_i'} \quad , i = 0, \dots, n.$$

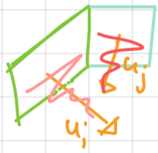
Combining this results in

$$\langle g(u_i), g(u_j) \rangle = \langle u_i', u_j' \rangle = \langle u_i, u_j \rangle. \Rightarrow g \text{ isometry.}$$ #

Suppose $C(\Theta)$ is positive definite. Then there ex. a square root $U = (u_0 | \dots | u_n) \in GL_{n+1}(\mathbb{R})$ of $C(\Theta)$ i.e.

$$C(\Theta) = U^T U.$$

Since $C_{ii}(\Theta) = 1$, u_i is unit vector, $i = 0, \dots, n$.



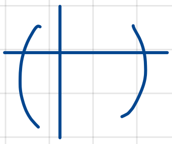
Moreover the halfspaces

$$\langle u_i, x \rangle \geq 0 \quad \text{and} \quad \langle u_j, x \geq 0 \qquad i,j = 0, \dots, n$$

have nonempty intersection (linear independent)

Thus, σ is a spherical simplex. ▨

Reminder



- The k^{th} - principal submatrix A_k of a matrix A is obtained by deleting the k^{th}-row and the k^{th}-column.
- $\langle x, y \rangle_{\mathbb{H}^n} := x_1 y_1 + \dots + x_n y_n - x_{n+1} y_{n+1}$ $\Rightarrow J = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$

Lemma H

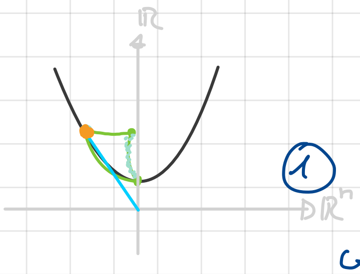
Let $C = (c_{ij}(\Theta))$ be a Gram matrix. Then σ is a hyperbolic simplex if and only if

1.) C is type $(n,1)$

2.) each principal submatrix of C is positive definite.

proof: "$\Rightarrow$"

Let C_k the the k-th principal submatrix of C and let σ by a hyperbolic simplex.



① Let $u_0, \dots, u_n$ be the unit inward-pointing units and $J = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & -1 \end{pmatrix}$ the matrix of $\langle \cdot, \cdot \rangle_{\mathbb{H}^n}$.

and $\quad U = (u_0 | \dots | u_n)$

Clearly

$$c_{ij}(\Theta) = U^T J U. \quad \Rightarrow \text{nondegenerate of type } (n,1).$$

(2.) Let $v_k \in \mathbb{H}^n$ be the vertex opposite to σ_k.

and

$$\underline{L_k} := \bigcap_{i \neq k} u_i^\perp$$

Since $\langle \cdot , \cdot \rangle_{\mathbb{H}^n}|_{L_k}$ is of type $(n,1)$ then $L_k^\perp$ is positive definite

Since $L_k^\perp$ spans A_k, A_k is positive definite.

"$\Leftarrow$" sketch:

Lemma:

Each hyperbolic simplex is uniquely determined by its Gram matrix. $\square$

There ex. U $\quad C = U^T J U \qquad U = (u_0 | \dots | u_n)$

Examine intersections $\langle u_i, x \rangle \geq 0$ ← simplicial cone

and show that $\langle u_i, x \rangle$ lies inside pos. light cone $\blacksquare$

Lemma E

Let $C = (c_{ij}(\Theta))$ be a Gram matrix. Then σ is a Euclidean simplex if and only if

1.) $c_{ij}(\Theta)$ is positive semidefinite of corank 1.

2.) $\ker(c_{ij}(\Theta)) = \langle v \rangle$ for some $v \in \mathbb{R}^{n+1}$, with positive coordinates $c_0, \dots, c_n$

proof: "$\Rightarrow$"

(1) Suppose σ is a Euclidean simplex and $C(\sigma)$ the Gram matrix. As before

$$\underset{\mathbb{R}^{n+1 \times n+1}}{\underset{\cap}{C(\sigma)}} = \mathcal{U}^T \mathcal{U} \qquad \mathcal{U} = (u_0 | \dots | u_n)$$

$$u_i \in \mathbb{R}^n \quad i = 1, \dots, n.$$

Since $\operatorname{span}\{u_i\}_{i=0}^{n} = \mathbb{R}^n$, $C(\sigma)$ has rank n and in particular $\operatorname{corank}(C(\sigma)) = 1$.
Moreover, $C(\sigma)$ is positive semidefinite.

(2) Note

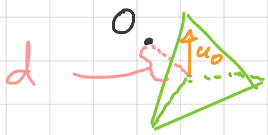
$$\ker(C(\sigma)) = \langle v \rangle, \quad v = \begin{pmatrix} c_0 \\ \vdots \\ c_n \end{pmatrix}$$

for some $c_i \in \mathbb{R}$ $i = 0, \dots, n$. with

$$c_0 u_0 + \dots + c_n u_n = 0 \iff v^T \mathcal{U}^T \mathcal{U} v = 0 \iff \mathcal{U} v = 0$$

$$\underline{c_i \geq 0 \quad ?}$$

Note that σ is defined by



$$\begin{aligned} \langle u_i, x \rangle &\geq 0 \qquad i = 1, \dots, n \\ \langle u_0, x \rangle &\geq -d . \end{aligned}$$

Then there ex. $v_i \in \mathbb{R}^n$ $i = 1, \dots, n$ s.t.

$$\langle u_j, v_i \rangle \underset{i \neq j}{=} 0 \quad \text{and} \quad \langle u_0, v_i \rangle = -d.$$

Take the inner product with v_i:

$$-c_0 d + c_i \langle u_i, v_i \rangle = 0$$

$$\iff \frac{\overbrace{\langle u_i, v_i \rangle}^{>0}}{\underbrace{d}_{>0}} = \frac{c_0}{c_i}$$

$\Rightarrow$ they have the same sign (taken to be positive)

"$\Leftarrow$" Let $\underset{\|}{\mathcal{U}}^{(u_0 | \dots | u_n)}$ be the sqcuare root of C (i.e. $\mathcal{U} \cdot \mathcal{U} = A^2$)
Moreover, describe $\mathcal{U}$ as follows:

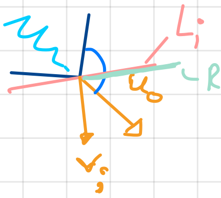
$$\begin{aligned} \mathcal{U} : \mathbb{R}^{n+1} &\longrightarrow \mathbb{R}^{n+1} \\ e_i &\longmapsto u_i \qquad \text{for } i = 1, \dots, n+1. \end{aligned}$$

Then $\ker(A) = \ker(\mathcal{U}) = \langle v \rangle$, $\operatorname{im}(\mathcal{U}) = v^{\perp}$

Hence they satisfy:

$$c_0 u_0 + \dots + c_n u_n = 0,$$

where $c_0, \dots, c_n$ are positive coefficients, non-zero.



Let C be a simplicial cone defined by

$$\langle u_0, x \rangle \geq -d.$$

Let R_i be the ray defined by $\langle u_i, x \rangle = 0$

Let L_i be the line defined by $\langle u_i, x \rangle = 0$

Let v_i be the unique point on L_i s.t. $\langle u_0, v_i \rangle = -d$

Then σ is a simplex if and only if $v_i \in R_i$

i.e. $\langle u_i, v_i \rangle \overset{(!)}{>} 0 \qquad i = 0, \dots, n.$

As in (1)

$$\langle u_i, v_i \rangle = \frac{c_0 d}{c_i}$$

But since c_0 and c_i have the same sign

$\langle u_i, v_i \rangle > 0$. ∎

Remarks

Condition 2.) holds automatically if all dihedral angles are non-obtuse

Why?

Lemma:

$$\sum_{i=0}^{n} A^i > 0$$

Let $A \in \mathbb{R}^{n \times n}$ indecomposable, $\nexists$ perturbation matrix P s.t. $PAP^T = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$, symmetric and positive semidefinite.

Then

If A is degenerate, then $\operatorname{corank}(A) = 1$ and it is spanned by a vector with coefficients >0.

Definition: cosine matrix

Suppose M is the Coxeter matrix of a Coxeter system on a set I. Then the cosine matrix $C = (c_{ij})_{i,j\in I}$ is defined by

$$c_{ij} := -\cos\left(\frac{\pi}{m_{ij}}\right) \qquad i,j \in I.$$

Convention:

$$m_{ij} = \infty \quad \Rightarrow \quad -\cos\left(\frac{\pi}{\infty}\right) = -\cos(0) = -1$$

Proposition L

Let M be the Coxeter matrix to an associated Coxeter group W on a set I and suppose no $m_{ij} = \infty$. Then W can be represented as a ...

(i) ... spherical reflection group generated by a spherical simplex $\Leftrightarrow$ the Gram matrix C is positive definite

Suppose M irreducible

(ii) ... Euclidean reflection group generated by a Euclidean simplex $\Leftrightarrow$ the Gram matrix C is positive semidefinite of corank 1

(iii) ... hyperbolic reflection group generated by a hyperbolic simplex $\Leftrightarrow$ the Gram matrix C is non-degenerate of type $(n,1)$ and each principal submatrix is positive definite.

proof: Combine Lemma S, E, H with the Main Thm. from last week. ☒

proof of Lannér's Thm (1)

Let $W \curvearrowright \mathcal{U}(W,\Delta^n)$ proper with fundamental domain an n-simplex. Acting proper on $\mathcal{U}(W,\Delta^n)$ is equivalent to the fact that the mirror structure on Δ^n is W-finite.

But W finite $\overset{(!)}{\Leftrightarrow}$ Cosine matrix C is positive definite (each principal submatrix.

$\overset{(!)}{\Leftrightarrow}$ finding all Coxeter diagrams Γ s.t. each proper sub diagram is positive definite

1.) $\det C > 0$, $\Rightarrow$ C positive definite

2.) $\det C = 0$, $\Rightarrow$ C positive semidefinite of corank 1

3.) $\det C < 0$, $\Rightarrow$ C is of type $(n,1)$

Apply Proposition L. ∎

Hyperbolic reflection groups in dimension 3

W geometric reflection group on:

(!) $\mathbb{S}^n$, $\mathbb{E}^n$, $\mathbb{H}^n$ $\Rightarrow$ fundamental domain: simplex / product of simplices / ?, simple

If fundamental domain is a simplex, how does P^n look like? ($m=2 \rightsquigarrow m$-gon, $m \geq 3$?)

m=3:

combinatorial equivalence: Two polytopes are combinatorially equivalent, if their set of faces are isomorphic

combinatorial type: An equivalence class of polytopes under combi. equivalence

Andreev's Theorem (combinatorial type)

Let P^3 be a simple polytope, E the edge set,

$$\Theta : E \longrightarrow (0, \tfrac{\pi}{2})$$

an angle assignment function.

Then (P^3, Θ) is a fundamental polytope of a hyperbolic reflection group $W \leq \mathrm{Isom}(\mathbb{H}^3)$ if and only if:

A1.) At each vertex we have

$$\Theta(e_1) + \Theta(e_2) + \Theta(e_3) > \pi$$

A2.) If 3 faces intersect but don't have a common vertex, then

$$\Theta(e_1) + \Theta(e_2) + \Theta(e_3) < \pi$$

A3.) 4 faces can't intersect perpendicular (i.e. $\Theta(e_i) = \frac{\pi}{2}$) unless two of the opposite faces also intersect

A4.) If P^3 is a triangular prism, then the angles along the base and top can't be all $= \frac{\pi}{2}$.

Moreover, W is unique up to isometry (in $\mathbb{H}^3$).

Uniqueness Proposition:

A simple convex polytope in $\mathbb{H}^3$ is determined up to isometry by its dihedral angles.

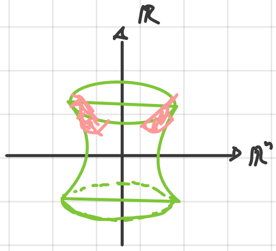
proof-strategy:

- show statement for each 2-dimensional face of P^3
- 2-dim faces of P^3 are determined up to concruence by the face angles.

↳ Cauchy's Geometric/Topological Lemma.

Definition: de Sitter sphere $\mathbb{S}^{n-1,1}$

$\mathbb{S}^{n-1,1} \subseteq \mathbb{R}^{n,1}$ n-dim hypersurface:



$$\mathbb{S}^{n-1,n} := \{ u \in \mathbb{R}^{n,1} \mid \langle u, u \rangle_{\mathbb{H}^n} = 1 \}$$

proof-sketch

Let $P^3 \subseteq \mathbb{H}^3$ be a simple polytope, $\tilde{\mathcal{F}}$ its set of faces.

For each $F \in \tilde{\mathcal{F}}$ let $u_F \in \mathbb{S}^{2,1}$ be the inward-pointing unit normal vector.

Then $\{u_F\}_{F \in \tilde{\mathcal{F}}}$ determines P (Proposition)

Denote $\mathcal{Y} := \{u_F\}_{F \in \tilde{\mathcal{F}}} \subseteq \mathbb{S}^{n-1,1}$.

Idea:

Identify: $I(P) := \Big\{$ "space of isometry classes of hyperbolic polytopes combinatorally equivalent to P". $\Big.$ $\simeq \mathcal{Y}/\mathrm{Isom}(\mathbb{H}^3)$

Moreover: $\mathcal{Y}/\mathrm{Isom}(\mathbb{H}^3)$ is a smooth manifold.

Then
$$\dim\left(\mathcal{Y}/\mathrm{Isom}(\mathbb{H}^3)\right) = f \cdot \dim(\mathbb{S}^{3-1,1}) - \dim(O(3,1)) = 3f - 6.$$

We set

$$C(P) := I(P) \cap \{\text{hyperbolic polytopes with dihedral angles} \leq \tfrac{\pi}{2}\}$$

As above we have

$$C(P) = 3f - 6.$$

We note that if we set

$$V(P) := \{ (0, \tfrac{\pi}{2}]^E \mid A1) - A4) \text{ are fulfilled} \},$$

and if we define

$$\Theta : C(P) \longrightarrow V(P)$$
$$Q \longmapsto (\theta(e))_{e \in E}.$$

dihedral angle of Q along e.

Then we need to show that Θ is a homeomorphism.

- injective: follows from uniqueness.
- Dimension count: $C(P)$ $\overset{(!)}{=}$ $V(P) \subseteq \mathbb{R}^E$

  $3f - 6$ (open $\Rightarrow$ e)

  Since $C(P) \subseteq I(P) \subseteq S^{2,1}$ we note that

  $\chi(S^2) = 2$, i.e. $3e - 3v + 3f = 3 \cdot 2 \iff 3f - 6 = 3e - 3v$ (*)

  Since 3 edges meet at each vertex we have

  $$3v = 2e.$$

  $$\overset{(*)}{\Rightarrow} \underset{\dim C(P)}{=} 3f - 6 = 3e - 2e = e = \dim(V)$$

- Θ is continuous (!) we can apply the theorem of the invariance of the domain that

  $$\Theta(C(P)) \subseteq V(P) \text{ open.}$$

$\rightarrow$ 'just' show that $\Theta(C(P)) = V(P)$!

$\blacksquare$