Coxeter systems and their combinatorial properties

Theorem

The following conditions on a pre-Coxeter system (W,S) are equivalent:

(i) (W,S) is a Coxeter system.

(ii) $\mathrm{Cay}(W,S)$ is a reflection system

(iii) (W,S) satisfies the exchange condition.

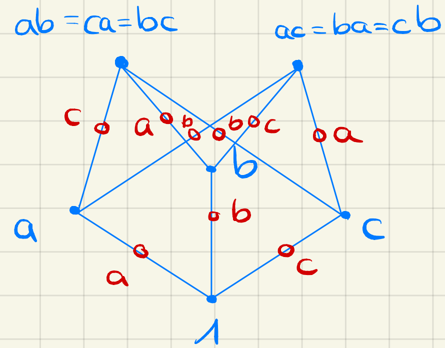
last talk

Examples / Motivation

(i) $W = \langle a, b, c \mid a^2 = b^2 = c^2 = 1,\ ba = ac,\ bc = ab \rangle$ is a pre-Coxeter system but <u>not</u> a Coxeter system.

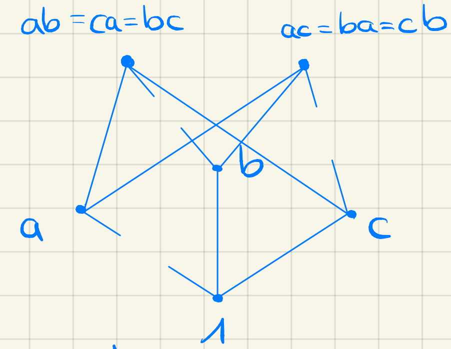
- Theorem (i) is not satisfied since the relations $ba = ac, bc = ab$ are not encoded in the Coxeter matrix.
- Theorem (ii) is not satisfied:

Cayley graph:



Ω

prereflection system



$\Omega \setminus \Omega^a$

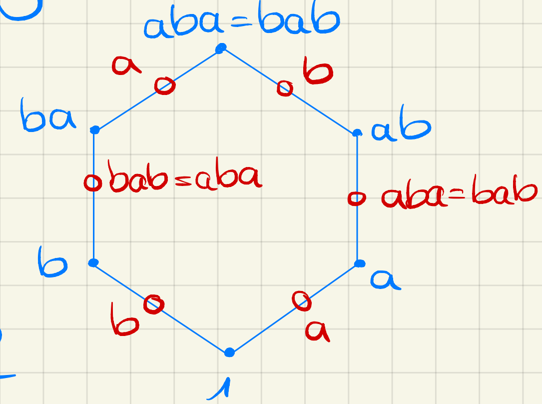
is still connected

- Theorem (iii) is not satisfied: ac is reduced and $cac = b$ so $\ell(cac) < \ell(ac)$, but $ac \neq ca$ and $ac \neq c^2 = 1$ so the Exchange condition is not satisfied.

(ii) $S_3 = \langle a, b \mid a^2 = b^2 = 1 , (ab)^3 = 1 \}$ is a Coxeter group (Dihedral group)

- Theorem (i) is clearly satisfied $\left(M = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \right)$
- Theorem (ii):

Cayley graph:

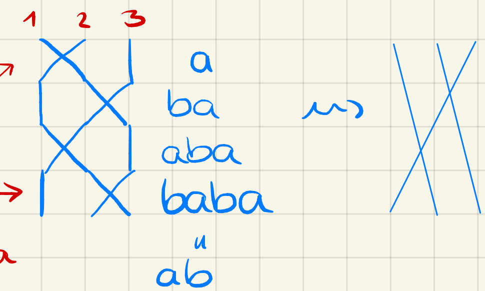


Ω

$\Rightarrow \Omega\setminus\Omega^{a}, \Omega\setminus\Omega^{b}, \Omega\setminus\Omega^{aba}$ have two components.

- Theorem (iii): „Wiring diagram" (example):

crosses wire 1 and 2



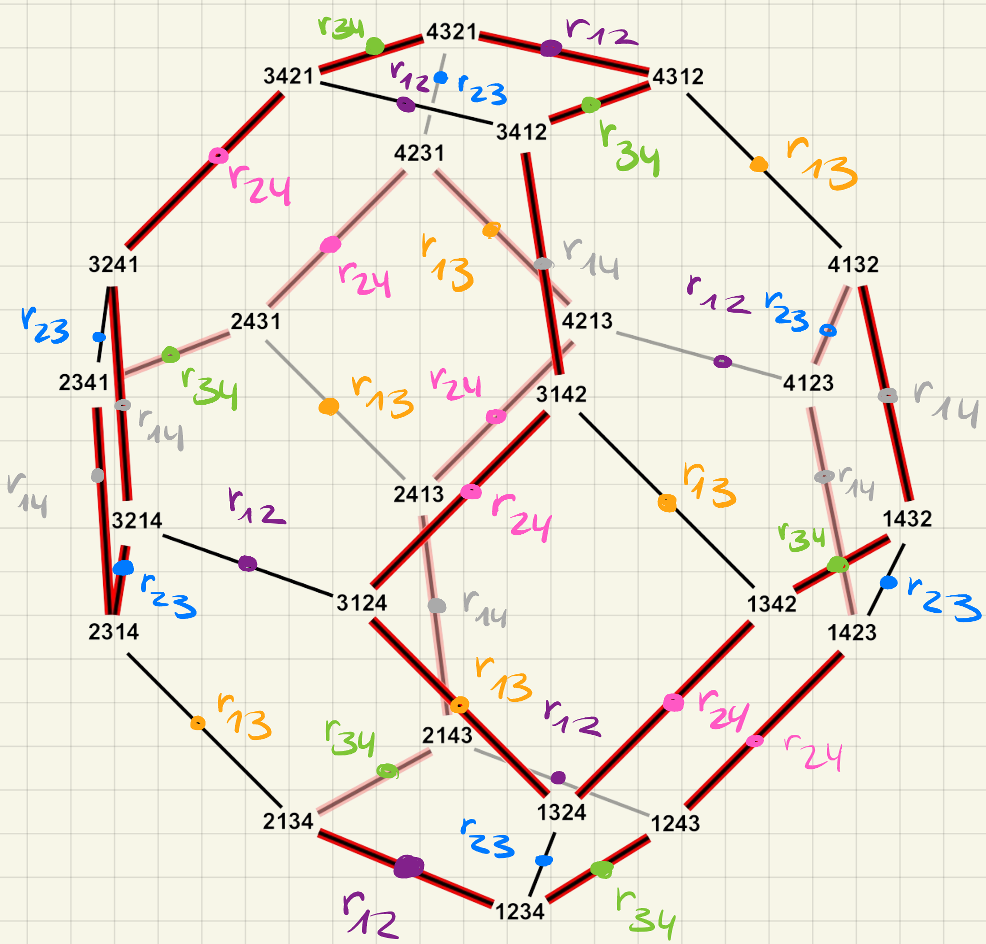
so $bab\cancel{a} = \cancel{b}aba$

(ii) $S_4 = \langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = 1, (s_1 s_2)^3 = (s_2 s_3)^3 = 1, (s_1 s_3)^2 = 1 \rangle$,

here $s_1 = (2134)$, $s_2 = (1324)$, $s_3 = (1243)$.

· Theorem (i) is clearly satisfied $M = \begin{pmatrix} 1 & 3 & 2 \\ 3 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}$.

· Theorem (ii) is satisfied:



· Theorem (iii) can be visualized as in the previous example.

Proof of (i) $\Rightarrow$ (ii): Coxeter group as signed permutations

Idea: For each Coxeter group W find a presentation in the group of permutations of the set $R \times \{\pm 1\}$.

- What's permuted? $\rightarrow$ The reflections in W.
- Define this presentation on generators first and show that it holds the relations.

Lemma

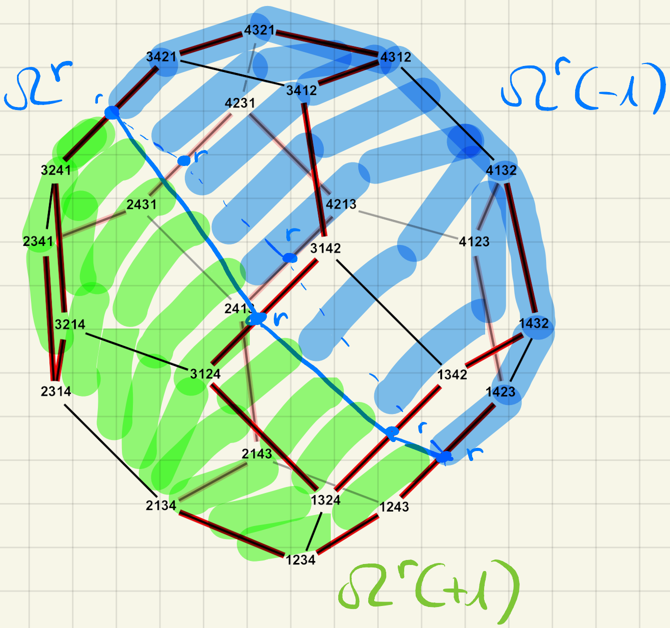
Suppose (W, S) is a Coxeter System.

(i) For any word s with $w = w(s)$ and any element $r \in R$ the number $(-1)^{n(r,s)}$ depends only on the value w. We denote this number by $\eta(r, w) \in \{\pm 1\}$.

(ii) There is a homomorphism $W \rightarrow \mathrm{Perm}(R \times \{\pm 1\})$, $w \mapsto \phi_w$ where the permutation ϕ_w is defined by the formula

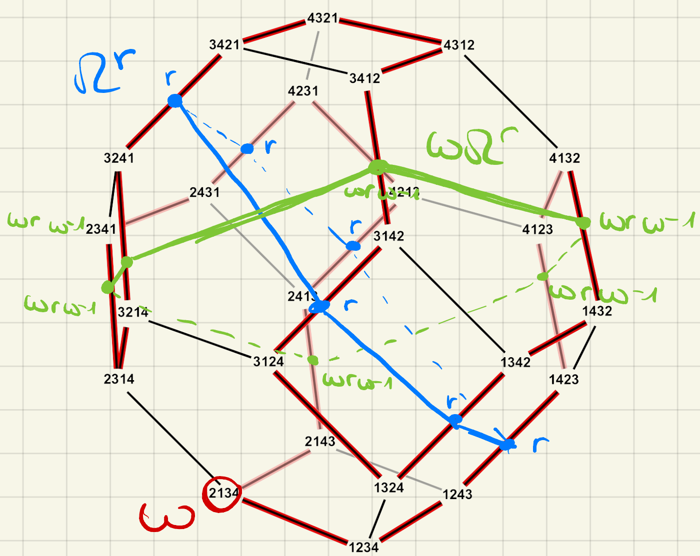
$$\phi_w(r, \varepsilon) = \left(w r w^{-1} , \eta(r, w^{-1}) \varepsilon \right)$$

Geometric idea:



→ $(-1)^{n(r,\omega)} = \begin{cases} +1, & 1,\omega \text{ on the same side of } \Omega^r \\ -1, & 1,\omega \text{ on the opposite side} \end{cases}$

→ the set of half-spaces $\Omega^r(\pm 1)$, $r \in R$ is indexed by $R \times \{\pm 1\}$, W acts on the set of half-spaces



Here $r = r_{24} = (1432)$
$wrw^{-1} = r_{14} = (4231)$
$w = (2134)$

→ For a given $w \in W$ the reflection across the wall $w\Omega^r$ is wrw^{-1}

$\Longrightarrow$ w maps $\Omega^r(+1)$ to $\Omega^{wrw^{-1}}(\varepsilon)$, $\varepsilon = \pm 1$

→ 1) $\varepsilon = +1 \Leftrightarrow w\Omega^r(+1) = \Omega^{wrw^{-1}}(+1)$
$\Leftrightarrow$ w and 1 are on the same side of $\overset{w \in w\Omega^r}{\overset{\|}{\Omega^{wrw^{-1}}}}$
$\Leftrightarrow$ w^{-1} and 1 are on the same side of Ω^r

2) $\varepsilon = -1 \Leftrightarrow \ldots \Leftrightarrow w^{-1}$ and 1 are on the opposite side of Ω^r

Proof of Lemma

For each $s \in S$ define

$$\phi_s : R \times \{\pm 1\} \longrightarrow R \times \{\pm 1\}, \quad \phi_s(r,\varepsilon) = \left(srs, \varepsilon(-1)^{\delta(s,r)}\right)$$

where $\delta(s,r)$ is the Kronecker delta.

- ϕ_s is a bijection (i.e. $\phi_s \in \mathrm{Perm}(R \times \{\pm 1\})$):

$$\phi_s^2(r,\varepsilon) = \phi_s\left(srs, \varepsilon(-1)^{\delta(s,r)}\right) = \left(ssrss, \varepsilon(-1)^{\delta(s,r)}(-1)^{\delta(s,r)}\right) = (r,\varepsilon)$$

Now let $\mathbf{s} = (s_1, \dots, s_k)$. Put $v = s_k \cdots s_1$ and $\phi_{\mathbf{s}} = \phi_{s_k} \circ \cdots \circ \phi_{s_1}$.

- claim: $\phi_{\mathbf{s}}(r,\varepsilon) = \left(vrv^{-1}, \varepsilon(-1)^{n(\mathbf{s},r)}\right)$
- proof by induction:
  - $\rightarrow k=1$: $\mathbf{s} = s_1 \Rightarrow \delta(s_1,r) = n(\mathbf{s},r)$ and $v = s_1 \Rightarrow vrv^{-1} = s_1 r s_1$.
  - $\rightarrow k>1$: $\mathbf{s}' = (s_1, \dots, s_{k-1})$, $u = s_{k-1} \cdots s_1$ suppose the claim holds for $\mathbf{s}'$.

  Then

$$\phi_{\mathbf{s}}(r,\varepsilon) = \phi_{s_k}\left(uru^{-1}, \varepsilon(-1)^{n(\mathbf{s}',r)}\right) = \left(vrv^{-1}, \varepsilon(-1)^{n(\mathbf{s}',r) + \delta(s_k, uru^{-1})}\right) \quad (*)$$

  Now

  $\Phi(\mathbf{s}) = (r_1, \dots, r_k)$, $r_k = s_1 \cdots s_{k-1} s_k s_{k-1} \cdots s_1 = u^{-1} s_k u$, so

  $\Phi(\mathbf{s}) = (\Phi(\mathbf{s}'), u^{-1} s_k u)$

  $\Rightarrow n(\mathbf{s},r) = \#(\text{times } r \text{ appears in } \Phi(\mathbf{s}))$

$$= n(s',r) + \delta(u^{-1}s_k u, r)$$

$$= n(s',r) + \delta(s_k, uru^{-1})$$

$$\Rightarrow \quad (*) = \left(vrv^{-1}, \varepsilon(-1)^{n(s,r)}\right). \quad \#$$

Therefore we have a map $S \to \mathrm{Perm}(R \times \{\pm 1\})$, $s \mapsto \phi_s$. In order to show that this extends to a homomorphism $W \to \mathrm{Perm}(R\times\{\pm 1\})$ we need to check the relations:

- We already know $(\phi_s)^2 = 1$.
- The other relations are of the form $(st)^m = 1$, $s,t \in S$, $m = m(s,t)$ order of st.

  So we need to check $1 \overset{!}{=} (\phi_{st})^m = (\phi_s \circ \phi_t)^m$

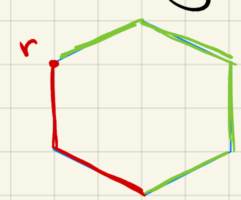
  1) clearly $(st)^m r (st)^m = r \quad \forall r \in R$ (1. component) ✓

  2) $s = \underbrace{(s,t,\dots,s,t)}_{m} \rightsquigarrow$ what's $n(s,r)$?

     1. case: $r \notin \langle s,t\rangle$ then $n(s,r) = 0$.

     2. case: $r \in \langle s,t \rangle \cong D_m$, then $n(s,r) = \#(\text{ways writing } r = \overbrace{st \cdots (s/t) \cdots ts}^{\text{maximal length } 2m-1}$.

     Since $r \in D_m$ there are exactly two ways of writing $r \neq 1$ in this form, so $n(s,r) = 2$.



$\Longrightarrow$ (ii) of the Lemma.

Now the definition of ϕ_w depends only on w so $(-1)^{\eta(r,s)}$ depends only on w what proofs (i). $\boxed{\not\!\!\!2}$

Define $\hat{R}_w$ as the set of all $r \in R$ s.t. $\eta(r,w) = -1$.

Remember $R(1,w) = \{ r \in R \mid \Omega^r \text{ separates } 1 \text{ from } w \}$

Proposition ((i) $\Rightarrow$ (ii) in the Theorem)

If (W,S) is a Coxeter system, then $Cay(W,S)$ is a reflection system. Moreover given $r \in R$ the vertices 1 and w lie on opposite sides of Ω^r if and only if $r \in \hat{R}_w$ (i.e. $\hat{R}_w = R(1,w)$).

Proof:

If s is a word for w then by the previous lemma each element of $\hat{R}_w$ occurs an odd number of times in $\Phi(s)$. In other words for each element $r \in \hat{R}_w$ any edge path in Ω from 1 to w must cross Ω^r an odd number of times and consequently 1 and w lie on opposite sides.

If $r \notin \hat{R}_w$ then there is an edge path connecting 1 to w which does not cross Ω^r.

$\Rightarrow$ $\Omega \setminus \Omega'$ has two components, $\hat{R}_\omega = R(1, \omega)$. $\boxtimes$

Proof of (iii) $\Rightarrow$ (i)   The word problem

Word problem: Suppose $\langle S | R \rangle$ is a presentation for a group G.
Question: Given a word $\mathbf{s}$ in $S \cup S^{-1}$ is there an algorithm for determining if its value $v(\mathbf{s})$ is the identity element of G?

→ for general groups there is no such algorithm.
→ for Coxeter groups we can solve this problem:

Suppose (W,S) is a pre-Coxeter system and $M = (m_{st})$ the associated Coxeter matrix.

Def
An elementary M-operation on a word in S is one of the following two types of operations:

(I) Delete a subword of the form (s,s)

(II) Replace an alternating subword of the form $(s,t,\dots)$ of length m_{st} by the alternating word $(t,s,\dots)$ of the same length m_{st}.

A word is M-reduced if it cannot be shortened by a sequence of elementary M-operations.

Theorem (Tits)

Suppose (W,S) satisfies the Exchange condition. Then

(i) A word s is a reduced expression if and only if it is M-reduced.

(ii) Two reduced expressions s and t represent the same element of W if and only if one can be transformed into the other by a sequence of elementary M-operations of type (II).

Proof

- proof of (ii): Let $s = (s_1, \dots, s_k)$, $t = (t_1, \dots, t_k)$ be reduced expressions for the same element. Induction on k:
  - $\to$ $k=1$: $s_1 = t_1 \Rightarrow$ words are the same.
  - $\to$ $k>1$: Set $s = s_1, t = t_1$.
    - 1. case: $s = t \Rightarrow (s_2, \dots, s_k), (t_2, \dots, t_k)$ are two reduced words of length $k-1$ for the same element $\overset{\text{Induction}}{\Rightarrow} (s_2, \dots, s_k) \overset{(\mathrm{II})}{\rightsquigarrow} (t_2, \dots, t_k) \Rightarrow s \overset{(\mathrm{II})}{\rightsquigarrow} t$.
    - 2. case: $s \neq t$. Put $m = m(s,t)$.
      - $\to$ claim: m is finite and we can find a third reduced expression u for w which begins with an alternating word $(s, t, \dots)$ of length m.
      - $\to$ Then: Let u' be the word obtained from u by the type (II) operation

that replaces the initial segment $(s, t, -)$ by $(t, s, \dots)$. Then we can transform:

$$s \overset{\text{case 1}}{\rightsquigarrow} u \overset{\text{above}}{\rightsquigarrow} u' \overset{\text{case 1}}{\rightsquigarrow} t$$

→ proof of claim: Let s_q, $q \geq 2$ be the alternating word in s and t of length q with final letter s.

$\Rightarrow$ s_q begins with s (q odd), s_q begins with t (q even).

We show by induction on q that we can find a reduced expression for w that begins with s_q for any $q \leq m$:

→ $q = 2$: We know $\ell(tw) < \ell(w)$ so apply (E):

$$w = t s_1 \cdots \hat{s}_i \cdots s_k$$

The exchanged letter cannot be s since $t \neq s$ so

$$w = t s s_2 \cdots \hat{s}_i \cdots s_k$$

is a reduced expression beginning with $(t, s) = s_2$

→ $q > 2$: Suppose we have such a word s' beginning with s_{q-1}. Let s' be the element of $\{s, t\}$ with which s_{q-1} does not begin. Since $\ell(s'w) < \ell(w)$ we can apply (E): We find another reduced expression for w by exchanging a letter of s' for an s' in front. The exchanged letter cannot be in the initial segment s_{q-1}

since in the dihedral group of order $2m$ a reduced expression for an element of length $\neq m$ is unique. (Induction step only works for $q < m$)

$\Rightarrow$ $\qquad w = s_q \cdots$ reduced expression.

This works for all $q \leq m$.

Since $q \leq \ell(w)$ we must have $m < \infty$ #claim 1 and w has a reduced expression beginning with $\mathfrak{s}_m$.

This reduced expression is either u (if m is odd) or u' (if m is even). We can replace $\mathfrak{s}_m$ by the other alternating word of length m to obtain the wished one. #(ii)

- <u>proof of (i)</u>:

"$\Rightarrow$": $\mathfrak{s}$ reduced $\Rightarrow$ $\mathfrak{s}$ M-reduced clear.

"$\Leftarrow$": $\mathfrak{s} = (s_1, \dots, s_k)$ M-reduced. Proof by induction on k:

→ $k = 1$: clear

→ $k > 1$: $\mathfrak{s}' = (s_2, \dots, s_k)$ is also M-reduced $\overset{\text{induction}}{\Rightarrow}$ $\mathfrak{s}'$ reduced.

Suppose: $\mathfrak{s}$ not reduced. Set $w = s_1 \cdots s_k$, $w' = \underbrace{s_2 \cdots s_k}_{\text{reduced}}$. We have

$\ell(s_1 w') = \ell(w) \leq k-1$. So apply ($\epsilon$):

w' has another reduced expression $\mathfrak{s}''$ beginning with s_1.

$\overset{(ii)}{\Rightarrow}$ We can transform $\mathfrak{s}' \rightsquigarrow \mathfrak{s}''$ by type (II) operations.

Thus s can be transformed by M-operations to a word beginning with (s_1, s_1) $\Rightarrow$ not M-reduced ↯

$\Rightarrow$ s must be reduced. $\boxed{}$

Proof of (iii) $\Rightarrow$ (ii) in the Theorem

(W, S) pre-Coxeter system that satisfies (E).
Now let $(\tilde{W}, \tilde{S})$ be the Coxeter system associated to the Coxeter matrix of (W, S) and let $p: \tilde{W} \to W$ be the natural surjection. In order to show that (W, S) is a Coxeter system we have to show that p is injective:
$\tilde{w} \in \ker(p)$, $\tilde{s} = (\tilde{s}_1, \dots, \tilde{s}_k)$ reduced expression for $\tilde{w}$.
$\Rightarrow$ $\tilde{s}$ M-reduced.
Let $s = (s_1, \dots, s_k)$ be the corresponding word in S. M-operations in S and $\tilde{S}$ are the same so s must be M-reduced as well
$\Rightarrow$ s reduced
Since s represents 1, s must be the empty word
$\Rightarrow$ $\tilde{s}$ is the empty word
$\Rightarrow$ $\tilde{w} = 1$. $\Rightarrow$ p injective. $\boxed{}$

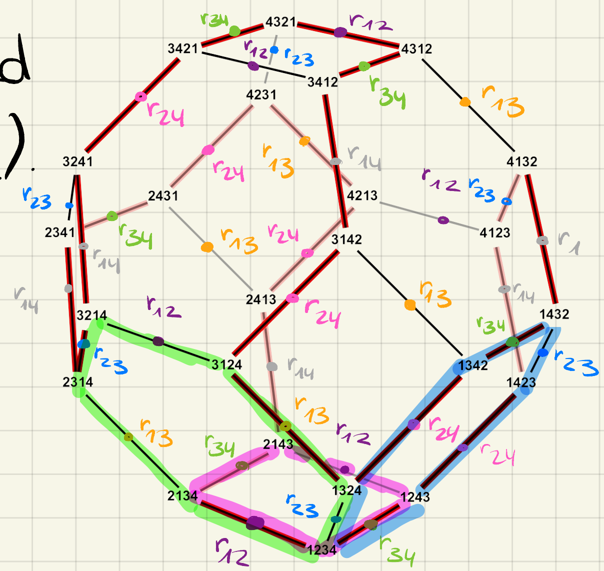
Special subgroups of a Coxeter group

Let (W, S) be a Coxeter system.

Def.
A *special subgroup* of W is one generated by a subset of S.
For each $T \subset S$, W_T denotes the subgroup generated by T.

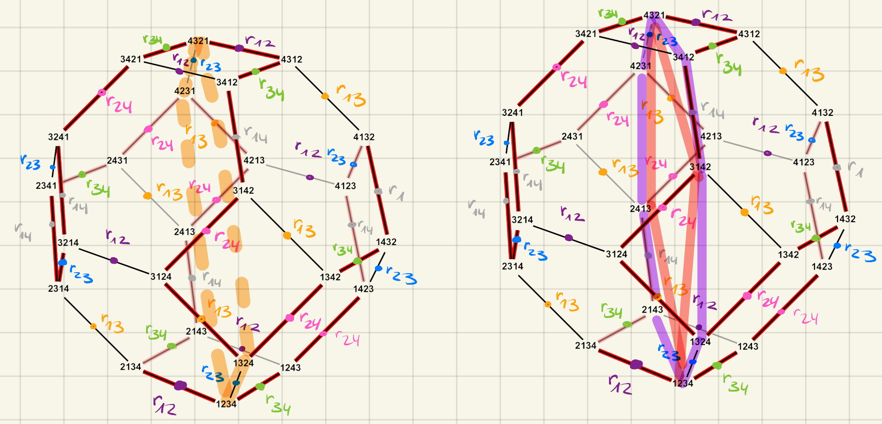
Example (special subgroups in S_4 generated by two elements)

- $W_{\{r_{12}, r_{34}\}}$ special subgroup generated by r_{12} and r_{34} (isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$).
- $W_{\{r_{12}, r_{23}\}}$ special subgroup generated by r_{12} and r_{23} (isomorphic to S_3)
- $W_{\{r_{23}, r_{34}\}}$ special subgroup generated by r_{23} and r_{34} (isomorphic to S_3)



Note:

- The subgroup generated by r_{14} and r_{23} is not a special subgroup but a Coxeter group
- The Cayley graph of a subgroup may not be a subgraph. For example take the subgroup generated by r_{23} and 2143 (isomorphic to dihedral group D_8)



- Not every subgroup must be a Coxeter group. For an example take the subgroup generated by 3142.

Proposition

For each $w \in W$, there is a subset $S(w) \subset S$ such that for any reduced expression $(s_1, \ldots, s_k)$ for w, $S(w) = \{s_1, \ldots, s_k\}$.

("The letters that can occur in a reduced expression just depend on the value of the word not the choice of reduced expression.")

Proof: Given two reduced expression s and s' for w we can transform

$s \rightsquigarrow s'$ by type(II) operations. This doesn't change the set of letters. $\boxtimes$

<u>Remark</u>: Let $(s_1, \dots, s_k)$ be a reduced expression for $w \in W$. Then $(s_k, \dots, s_1)$ is a reduced expression for w^{-1}. We get:

$$S(w^{-1}) = S(w) \quad \forall w \in W \qquad (*)$$

Given reduced expressions for $v, w \in W$ we can concatenate them to get a word for vw. By the Deletion property there are no new letters in a reduced expression for vw. So we get:

$$S(vw) \subseteq S(v) \cup S(w) \quad \forall v, w \in W \qquad (**)$$

<u>Corollary</u>

For each $T \subset S$, W_T consists of those elements $w \in W$ such that $S(w) \subset T$.

<u>Proof</u>: Define $X = \{ w \in W \mid S(w) \subset T \}$

- <u>$X \subseteq W_T$</u>: $w \in X$, $(s_1, \dots, s_k)$ reduced expression for w $\Rightarrow s_1, \dots, s_k \in T$

  $\Rightarrow s_1 \cdots s_k \in W_T$.
- <u>X is a subgroup of W</u>: $v, w \in X \Rightarrow S(vw) \overset{(**)}{\subseteq} S(v) \cup S(w) \subseteq T \Rightarrow vw \in X$.

$w \in X \Rightarrow S(w^{-1}) = S(w) \subseteq T \Rightarrow w^{-1} \in X.$

- Since $T \subseteq X$ we get $W_T \subseteq X$.

$\Rightarrow W_T = X.$ $\boxed{3}$

Corollary

For each $T \subset S$, $W_T \cap S = T$.

- $T \subseteq W_T \cap S$: clear
- $W_T \cap S \subseteq T$: $s \in W_T \cap S \Rightarrow s \in X \Rightarrow s \in S(s) \subset T \Rightarrow s \in T.$

Corollary

S is a minimal set of generators for W.

Suppose $T \subsetneq S$, $W_T = W$, let $s \in S \setminus T \subseteq W$. Then $s \in W \cap S = W_T \cap S = T \not\ni s$ ↯

Corollary

For each $T \subset S$ and each $w \in W_T$ the length of w with respect to T (denoted by $\ell_T(w)$) is equal to the length of w with respect to S (denoted by $\ell_S(w)$).

Proof: Let $(s_1, \ldots, s_k)$ be a reduced expression for $w \in W_T$ in W. Now we have $S(w) \subset T \Rightarrow s_1, \ldots, s_k \in T \Rightarrow \ell_T(w) = k = \ell_S(w).$ $\boxed{2}$

Theorem

(i) For each $T \subset S$, (W_T, T) is a Coxeter system.

(ii) Let $(T_i)_{i \in I}$ be a family of subsets of S. If $T = \bigcap_{i \in I} T_i$ then

$$W_T = \bigcap_{i \in I} W_{T_i}.$$

(iii) Let T, T' be subsets of S and w, w' elements of W. Then $wW_T \subset w'W_{T'}$ (resp. $wW_T = w'W_{T'}$) if and only if $w^{-1}w' \in W_{T'}$ and $T \subset T'$ (resp. $T = T'$).

Proof:

- for (i): (W_T, T) is a pre-Coxeter system. So it suffices to show that it satisfies the Exchange condition:

  Let $t \in T$, $w \in W_T$ such that $\ell_T(tw) \leq \ell_T(w) \Rightarrow \ell_S(tw) \leq \ell_S(w)$.

  Let $\underline{t} = (t_1, \dots, t_k)$, $t_i \in T$ be a reduced expression for w. Since W satisfies the Exchange condition a letter of $\underline{t}$ can be exchanged for a t in front.

  Hence (W_T, T) satisfies (E). #

- for (ii): $W_{T_i} = \{w \in W \mid S(w) \subseteq T_i\} =: X_i$, $W_T = \{w \in W \mid S(w) \subseteq T\}$

$$\Rightarrow \bigcap_{i\in I} W_{T_i} = \bigcap_{i\in I} \{w\in W \mid S(w)\subseteq T_i\} = \{w\in W \mid S(w)\subseteq \underbrace{\bigcap_{i\in I} T_i}_{=T}\}$$

$$= W_T \;\#$$

· for (iii):

"$\Rightarrow$": $wW_T \subset w'W_{T'} \Rightarrow w'^{-1}wW_T \subset W_{T'} \overset{1\in W_T}{\Rightarrow} w'^{-1}w \in W_{T'} \Rightarrow w^{-1}w' \in W_{T'}$

$\Rightarrow W_T \subset W_{T'} \overset{\text{Corollary}}{\Rightarrow} T \subseteq T'$.

"$\Leftarrow$": $T\subseteq T' \overset{\text{Corollary}}{\Rightarrow} W_T \subseteq W_{T'}$, $w^{-1}w' \in W_{T'} \Rightarrow W_{T'} = w^{-1}w'W_{T'}$

$\Rightarrow W_T \subseteq w^{-1}w'W_{T'} \Rightarrow wW_T \subseteq w'W_{T'}$. $\boxed{}$

Proposition

Suppose S can be partitioned into two nonempty disjoint subsets S', S'' such that $m_{st} = 2$ $\forall s\in S', t\in S''$. Then $W = W_{S'} \times W_{S''}$.

Proof:
- $S' \cup S'' = S$ generates W $\Rightarrow$ every element $w\in W$ is of the form $w = w'\cdot w''$ for $w'\in W_{S'}$, $w''\in W_{S''}$.
- Suppose $w\in W_{S'}\cap W_{S''}$ $\Rightarrow$ $w = s_1\cdots s_k$ with $s_i \in S'\cap S''$ $\Rightarrow$ $w = 1$
- $w'\in W_{S'}$, $w''\in W_{S''}$ $\Rightarrow$ $w' = s_1' - s_k'$, $s_i'\in S'$, $w'' = s_1'' \cdots s_l''$, $s_i''\in S''$

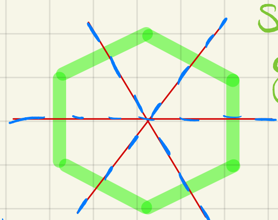
$$\Rightarrow \omega'\omega'' = s'_1 - s'_k s''_1 - s''_e = s'_1 - s'_{k-1} s''_1 s''_2 - s''_e s'_k$$
$$= \ldots = s''_1 - s''_e s'_1 - s'_k$$
$$= \omega''\omega' . \quad \square$$

Remark

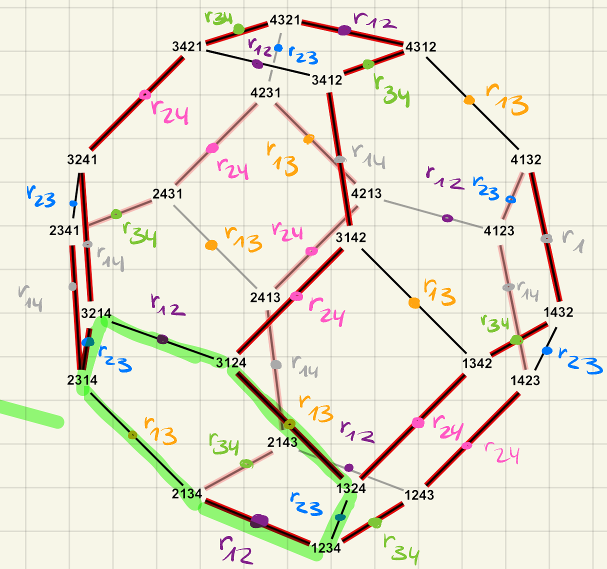
In a special subgroup there are two possible definitions for reflections

1) reflections from the original group (conjugates of elements in S that lie in W_T)

2) reflections contained in the subgroup (conjugates of elements in T)



Special subgroup generated by r_{12} and r_{23}



The following lemma shows that these actually coincide.

Lemma

Suppose T is a subset of S. If $r \in R \cap W_T$ then there is an element $w \in W_T$ such that $w^{-1} r w \in T$.

Another characterization of Coxeter systems

Proposition

Suppose (W,S) is a pre-Coxeter system and $\{P_s\}_{s\in S}$ family of subsets satisfying the following conditions:

(A) $1 \in P_s \quad \forall s \in S$

(B) $P_s \cap sP_s = \emptyset \quad \forall s \in S$

(C) Suppose $w \in W$ and $s,t \in S$. If $w \in P_s$ and $wt \notin P_s$ then $sw = wt$.

Then (W,S) is a Coxeter system and $P_s = \{w \in W \mid \ell(sw) > \ell(w)\}$

Proof: Suppose $s \in S$ and $w \in W$. There are two possibilities:

1. case: $w \notin P_s$. Let $s_1 \cdots s_k$ be the reduced expression for w and let $(w_0, \dots, w_k)$ be the corresponding edge path $w_0 = 1$, $w_i = s_1 \cdots s_i = w_{i-1}s_i$ $1 \le i \le k$.

$\overset{(A)}{\Longrightarrow} w_0 \in P_s, w_k \notin P_s \Rightarrow \exists\, i \in \{1, \dots, k\}$ s.t. $w_{i-1} \in P_s, w_i \notin P_s$

$\overset{(C)}{\Longrightarrow} sw_{i-1} = w_{i-1}s_i \Longrightarrow sw = ss_1 \cdots s_k = sw_{i-1}s_i \cdots s_k = s_1 \cdots \hat{s}_i \cdots s_k$ and

$\ell(sw) < \ell(w) \Longrightarrow$ Exchange condition holds $\Rightarrow$ Coxeter system.

2. case: $w \in P_s$. Then $w' := sw \overset{(B)}{\Longrightarrow} w' \notin P_s$

$\overset{1.\text{ case}}{\Longrightarrow} \ell(sw) = \ell(w') > \ell(sw') = \ell(w) \Longrightarrow$ Exchange condition holds

$\Rightarrow$ Coxeter System. $\square$