CAT(κ)-spaces, cell complexes and the link condition

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May 21, 2021

Geodesics

A geodesic in a metric space (X, d) is an isometric embedding $c : [a, b] \longrightarrow X$, where $[a, b] \subseteq \mathbb{R}$ is a closed interval. A metric space X is called geodesic if for all $p, q \in X$ there is a geodesic from p to q.

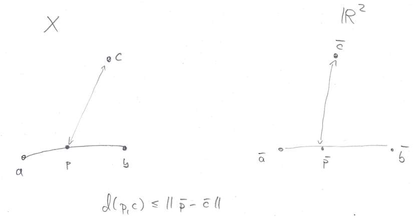
Examples

- Every normed real vectorspace is geodesic.
- Every connected metric graph is geodesic.
- The metric d_g of every complete Riemannian manifold (M, g) is geodesic.

[Note that a Riemannian geodesic, which is defined by $\frac{\nabla}{dt}\dot{c} = 0$, is not exactly the same as a geodesic as defined above. However, every geodesic in our sense in a Riemannian manifold is a Riemannian geodesic.]

A definition of CAT(0)-spaces

A (nonempty) geodesic metric space (X, d) is called a CAT(0)-space if the following holds. Let $a, b, c \in X$. By the triangle inequality, there exist points $\bar{a}, \bar{b}, \bar{c}$ in euclidean space $\mathbb{R}^2$ with $d(a, b) = \|\bar{a} - \bar{b}\|$, $d(b, c) = \|\bar{b} - \bar{c}\|$, $d(c, a) = \|\bar{c} - \bar{a}\|$. If p is a point on a geodesic from a to b, and if $\bar{p}$ is the corresponding point on the line from a to b, with $d(a, p) = \|\bar{a} - \bar{p}\|$ and $d(p, b) = \|\bar{p} - \bar{b}\|$, then $d(p, c) \leqslant \|\bar{p} - \bar{c}\|$.



Examples

- A normed vectorspace $(V, |.|)$ is CAT(0) if and only if the norm satisfies the parallelogram law, $|u - v|^2 + |u + v|^2 = 2(|u|^2 + |v|^2)$.
- A metric graph is a CAT(0)-space if and only if it is a tree.
- Every complete, simply connected Riemannian manifold of sectional curvature $\leqslant 0$ is a CAT(0)-space.

Properties of CAT(0)-spaces

- Geodesics in CAT(0)-spaces are unique.
- The cartesian product of CAT(0)-spaces is again a CAT(0)-space.
- A nonempty geodesically convex subset of a CAT(0)-space is again a CAT(0)-space.
- If $f : X \longrightarrow Y$ is a locally isometric map between CAT(0)-spaces, then f is an isometric embedding.

Theorem

Let X be a CAT(0)-space, let $A \subseteq X$ be a complete, convex and nonempty subset. Then for every $p \in X$, there is a unique point $\pi_A(p) \in A$ at minimal distance from p. The map $\pi_A : X \longrightarrow A$ is 1-Lipschitz and a strong deformation retraction. In particular, X is contractible.

Theorem (Bruhat-Tits)

Let X be a complete CAT(0)-space and let $B \subseteq X$ be bounded. Then there is a unique point $p \in X$ and $r \geqslant 0$ such that $\bar{B}_r(p)$ is the smallest closed ball containing B. The point p is called the center of B.

Corollary (Bruhat-Tits Fixed Point Theorem)

Let X be a complete CAT(0)-space and let G be a group that acts isometrically on X. If some $q \in X$ has a bounded G-orbit $G(q)$, then G has a fixed point in X.

Angles

Let $c_1, c_2 : [0, r] \longrightarrow X$ be geodesics in a CAT(0) space, with $r > 0$ and $p = c_1(0) = c_2(0)$. The angle $\angle(c_1, c_2) \in [0, \pi]$ is defined by

$$\sin(\frac{1}{2}\angle(c_1, c_2)) = \lim_{t \to 0} \frac{d(c_1(t), c_2(t))}{2t}.$$

The angle defines a pseudometric on the set of all non-constant geodesics starting at p. The metric completion of this pseudometric space is the space of directions $\Sigma_p X$.

In a Riemannian manifold M, the space of directions can be identified with the unit sphere in the tangent space $T_p M$.

The spherical distance between two points $a, b \in \mathbb{S}^2$ is given by $\cos(d_{\mathbb{S}^2}(a, b)) = \langle a, b \rangle$.

The CAT(1) condition

Let Σ be a metric space. Given $a, b, c \in \Sigma$ with $d(a, b) + d(b, c) + d(c, a) < 2\pi$, there exists points $\bar{a}, \bar{b}, \bar{c} \in \mathbb{S}^2$ with $d(a, b) = d_{\mathbb{S}^2}(\bar{a}, \bar{b})$, $d(b, c) = d_{\mathbb{S}^2}(\bar{b}, \bar{c})$, $d(c, a) = d_{\mathbb{S}^2}(\bar{c}, \bar{a})$. We call Σ a CAT(1) space if the following hold in this situation.

- There is a geodesic from a to b.
- If p is a point on some geodesic from a to b, with comparison point $\bar{p} \in \mathbb{S}^2$, then $d(p, c) \leqslant d_{\mathbb{S}^2}(\bar{p}, \bar{c})$.

Examples

- The sphere $\mathbb{S}^n$, with the angular metric, is a CAT(1)-space.
- Every CAT(0)-space is a CAT(1)-space.
- A normed vectorspace is a CAT(1)-space if and only if the norm satisfies the parallelogram law.
- A metric graph is a CAT(1)-space if it contains no circles of length $< 2\pi$.
- Every complete, simply connected Riemannian manifold of sectional curvature $\leqslant 1$ is a CAT(1)-space.
- In a CAT(1) space, geodesics between points at distance $< \pi$ are unique.
- If X is a CAT(0)-space and if $p \in X$, then the space of directions $\Sigma_p X$ is a CAT(1)-space.

Convex polytopes

A convex polytope P is the convex hull of a finite set of points in some real vector space (possibly of infinite dimension). If an affine hyperplane H intersects P nontrivially, and if P is contained in one of the two half-spaces determined by H, then $F = P \cap H$ is called a face of P. Then F is again a convex polytope, and P has finitely many faces.

Examples

Let $e_1, \dots, e_n$ be the standard basis of $\mathbb{R}^n$. The convex hull of $e_1, \dots, e_n$ is called the standard $n-1$-simplex Δ^{n-1}. The convex hull of $\pm e_1, \dots, \pm e_n$ is called the standard n-octahedron $\diamond^n$. The convex hull of $\pm e_1 \pm e_2 \cdots \pm e_n$ is called the standard n-cube $\square^n$.

Convex cell complex

A convex cell complex is a collection $\mathcal{C}$ of convex polytopes in some real vector space W, with the following properties.

- If P is in $\mathcal{C}$ and if F is a face of P, then F is in $\mathcal{C}$.
- If $\mathrm{P}, \mathrm{Q} \in \mathcal{C}$ have nonempty intersection, then $\mathrm{P} \cap \mathrm{Q}$ is a face in P and in Q.

Every polytope P carries a natural compact topology (and is homeomorphic to a closed ball in some euclidean space). The weak topology on $\mathrm{X}(\mathcal{C}) = \bigcup \mathcal{C}$ is defined as follows. A set $A \subseteq \mathrm{X}(\mathcal{C})$ is closed if and only if $A \cap \mathrm{P}$ is closed in P, for every $\mathrm{P} \in \mathcal{C}$. In the weak topology, $\mathrm{X}(\mathcal{C})$ is a regular CW complex (but possibly not metrizable).
[Bridson-Haefliger use a slightly more general definition.]

Abstract simplicial complexes

An abstract simplicial complex is a collection S of finite sets with the following property:

- If $a \subseteq b \in S$, then $a \in S$.

The elements of S are called simplices, and the elements of the singletons in S are called vertices.

Example

Let $(P, \leqslant)$ be a poset. The derived complex P' consists of all linearly ordered finite subsets of P. This is an abstract simplicial complex.

The geometric realization of an abstract simplicial complex

Let S be an abstract simplicial complex with vertex set V. Let W be the real vector space with basis V. The elements of W are thus finite formal linear combinations of vertices. If $\{v_0, \ldots, v_k\}$ is a simplex, we let P denote the convex hull of $\{v_0, \ldots, v_k\}$ in W. The resulting convex cell complex is denoted by $\mathcal{C}(S)$, and $X(\mathcal{C}(S))$ is called the geometric realization of S.

Barycentric subdivision

Let $\mathcal{C}$ be a convex cell complex. Then $\mathcal{C}$ is, in particular, a poset. The derived complex $\mathcal{C}'$ is called the barycentric subdivision of $\mathcal{C}$. If we choose for each cell $P \in \mathcal{C}$ a point c_P in the interior of P (eg. the center of mass of P), then we have a bijection from the vertices of the derived complex $\mathcal{C}'$ to the points $\{c_p \mid P \in \mathcal{C}\}$, which extends linearly to a bijection

$$X(\mathcal{C}') \longrightarrow X(\mathcal{C}).$$

With respect to the weak topologies, this map is a homeomorphism.

Piecewise euclidean cell complexes

Suppose that $\mathcal{C}$ is a convex cell complex, and that for every $P \in \mathcal{C}$, we fix an affine linear bijection $i_P : P \longrightarrow P'$ to a convex polytope P' in some finite dimensional euclidean vector space. Then i_P induces a metric d_P on P. We call such a collection of affine linear bijections compatible if the metrics induced by i_P and i_Q agree on $P \cap Q$, for all $P, Q \in \mathcal{C}$. We then call $\mathcal{C}$ with the collection of metrics $d = \{d_P \mid P \in \mathcal{C}\}$ a piecewise euclidean cell complex.

The pseudometric

Let $(\mathcal{C}, d)$ be a piecewise euclidean cell complex. A string in $X(\mathcal{C})$ is a sequence of points $x_0, \ldots, x_m$ in X such that x_j, x_{j-1} are contained in a common polytope P_j, for $j = 1, \ldots, m$. The length of the string is then $\ell(x_0, x_m) = \sum_{j=1}^{m} d_{P_j}(x_{j-1}, x_j)$. The distance $d_\ell(p, q)$ is defined to be the infimum of the lengths of all strings from p to q. It is clear that d_ℓ is a pseudometric on X.

• In general, d_ℓ will not be a metric. Let $\mathcal{C}$ be the metric graph with vertex set $\mathbb{N} \cup \{\pm\infty\}$, with edges of length 2^{-n} between n and $\pm\infty$. In the resulting pseudometric, $d_\ell(-\infty, +\infty) = 0$.
• If $\mathcal{C}$ is not locally finite, then the weak topology on $X(\mathcal{C})$ is not metrizable. Hence d_ℓ will possibly induce a different topology on $X(\mathcal{C})$.

Theorem (Bridson)

Let $(\mathcal{C}, d)$ be a piecewise euclidean cell complex. If $\mathcal{C}$ is locally finite or if there are only finitely many isometry types of polytopes in $\mathcal{C}$ (then we say that $(\mathcal{C}, d)$ has finitely many shapes), then d_ℓ is a complete metric on $X(\mathcal{C})$.
Points at finite distance can be joined by geodesics.
For every $x \in X(\mathcal{C})$, there is an $\varepsilon_x > 0$ such that the following holds.
• If $d_\ell(x, y) < \varepsilon_x$, then $x, y \in P$ for some $P \in \mathcal{C}$, and $d_P(x, y) = d_\ell(x, y)$.

In a similar way, one may define piecewise spherical cell complexes. In this case, the metric on the polytopes is induced by metrics on convex polyhedral subsets of spheres. An analog of Bridson's Theorem holds for such piecewise spherical cell complexes.

Theorem (Dowker)

Let $(\mathcal{C}, d)$ be a piecewise euclidean cell complex. Assume that $\mathcal{C}$ is locally finite or that there are only finitely many isometry types of polytopes in $(\mathcal{C}, d)$. Then the identity map is a homotopy equivalence between the weak topology on $X(\mathcal{C})$ and the topology determined by d_ℓ.

The geometric link in a convex euclidean polytope

Let P be a convex polytope in euclidean space, and let $p \in P$. The inward tangent cone $C_p P$ of P at p consists of all vectors v such that $p + tv \in P$ holds for some $t > 0$. This set is a closed convex cone in the ambient vector space. The geometric link $lk_p(P)$ is the set of all unit vectors in $C_p P$. This set is a convex spherical polytope, which we endow with the angular metric.

If p is an interior point in P, then $lk_p P$ is a sphere of dimension $\dim(P) - 1$.

The geometric link

Suppose that $(\mathcal{C}, d)$ is a piecewise euclidean cell complex. The geometric link $\mathrm{lk}_p X$ of $p \in X(\mathcal{C})$ is the union of all links $\mathrm{lk}_p(P)$, for $p \in P \in \mathcal{C}$. This set is a piecewise spherical cell complex in a natural way.

If $(\mathcal{C}, d)$ has finitely many shapes, then the same is true for $\mathrm{lk}_p X$.

By Bridson's Theorem for piecewise spherical complexes, we obtain a complete metric d_{lk} on $\mathrm{lk}_p X$, where points at finite distance can be joined by a geodesic.

Theorem (Gromov, Ballmann, Bridson)

Let $(\mathcal{C}, d)$ be a piecewise euclidean cell complex with finitely many shapes. The following are equivalent.

- $(X(\mathcal{C}), d_\ell)$ is a CAT(0) space.
- $(X(\mathcal{C}), d_\ell)$ is simply connected and each $\mathrm{lk}_p X$ is a CAT(1)-space for every vertex $\{p\} \in \mathcal{C}$.
- $(X(\mathcal{C}), d_\ell)$ is contractible and each $\mathrm{lk}_p X$ is a CAT(1)-space for every vertex $\{p\} \in \mathcal{C}$.
- $(X(\mathcal{C}), d_\ell)$ is uniquely geodesic and each $\mathrm{lk}_p X$ is a CAT(1)-space for every vertex $\{p\} \in \mathcal{C}$.

Literature:

Bridson, Haefliger, Spaces of non-positive curvature.
Davis, The geometry and topology of Coxeter groups.