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Winter term 2024/25
Universität Münster

Locally Compact Groups
Sheet 10

Hand in: Friday December 20, 2024 after class in letterbox No 4.

Problem 1.
Let $(E, ||\cdot||)$ be a Banach space and let $K \subseteq E$ be a closed subset. Show (directly) that K is compact if and only if for every $\varepsilon > 0$, the set K is contained in a union of finitely many ε-balls.

Problem 2.
Give an example of an unbounded linear operator on a normed vector space.

Problem 3.
Let $E = C([0,1], \mathbb{R})$ with the norm $||f||_\infty = \max\{|f(t)| \mid t \in [0,1]\}$. Show that E is a Banach space. Show that the operator $T : E \to E$ defined by $(Tf)(x) = \int_0^x f(t)dt$ is linear and bounded. Compute $||T||$ and $\ker(T)$. Is T an open map?

Problem 4.
Let E, F, H be normed vector spaces and let $S : E \to F$ and $T : F \to H$ be bounded operators. Show that $||T \circ S|| \leq ||T|| \cdot ||S||$.

Bonus Problem 1.
Let E be a Banach space and let $T \in B(E, E)$, with $||T|| < 1$. Show that $R = \sum_{k=0}^{\infty} T^k$ exists, that $R \in B(E, E)$, and that $RT = TR = \mathrm{id}_E - R$.