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Locally Compact Groups

Sheet 8

Hand in: Friday December 6, 2024 after class in letterbox No 4.

Problem 1.
Give an example of a locally compact, non-discrete and countable space. Prove then that a countable locally compact group is discrete.

Problem 2.
Let $\mathrm{U}(1) = \{z \in \mathbb{C} \mid |z| = 1\}$ denote the circle group and put $z(t) = \exp(2\pi i t)$, where $i = \sqrt{-1}$. Show that $I(f) = \int_0^1 f(z(t))dt$ is a left invariant integral on $\mathrm{U}(1)$.

Problem 3.
Let F be a field, let G be a group and consider the group ring $F[G]$. Let $i : G \to F[G]$ denote the map that maps $g \in G$ to the element $1g \in F[G]$. Show that this map i has the following universal property: if V is an F-vector space and if $f : G \to \mathrm{GL}(V) \subseteq \mathrm{End}(V)$ is a homomorphism from the group G to the group of invertible linear endomorphisms of V, then there is a unique ring homomorphism $\bar{f} : F[G] \to \mathrm{End}(V)$ such that the diagram

$$\begin{array}{ccc} G & \xrightarrow{\ i\ } & F[G] \\ & \searrow^{f} & \downarrow{\bar f} \\ & & \mathrm{End}(V) \end{array}$$

commutes.

A *semigroup* $(S, \cdot)$ consists of a set S and an associative multiplication $\cdot$ on S. A *compact semigroup* is a semigroup which carries a compact topology such that $\cdot$ is continuous.

Problem 4.
Let S be a compact semigroup, with $S \neq \emptyset$.
(1) Show that for every $a \in S$, the set aS is a compact semigroup.
(2) Show that the set $\mathcal{C}$ of all nonempty compact subsemigroups contains minimal elements $T \in \mathcal{S}$, i.e. that there is a compact nonempty subsemigroup $T \subseteq S$ which is minimal with this property. [*Hint. Use Zorn's Lemma.*]
(3) **Ellis' Theorem.** Show that all elements of such a minimal T are idempotent, i.e. satisfy $t^2 = t$. In particular, conclude that S contains idempotents. [*Hint. Consider* $A = tT \subseteq T$ *and then* $B = \{r \in T \mid tr = t\} \subseteq T$.]
(4) Let G be a Hausdorff topological group, let $K \subseteq G$ be a nonempty compact subset which is a semigroup, i.e. $KK \subseteq K$. Show that K is a subgroup. [*Hint. For* $s \in K$ *consider* $S = sK$ *and use (3).*]

Bonus Problem 1.
The compact interval $[0, 1] \subseteq \mathbb{R}$ is under multiplication a compact semigroup, but not a group. Why does this not contradict Problem 8.4(4)?

Bonus Problem 2.
Let G be a group and let p be a prime. Suppose that every element $g \in G$ satisfies $g^p = e$. If $p = 2$, show that G is abelian. Can one draw the same conclusion if $p > 2$?