Locally Compact Groups Sheet 8

Hand in: Friday December 6, 2024 after class in letterbox No 4.

Problem 1.

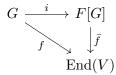
Give an example of a locally compact, non-discrete and countable space. Prove then that a countable locally compact group is discrete.

Problem 2.

Let $U(1) = \{z \in \mathbb{C} \mid |z| = 1\}$ denote the circle group and put $z(t) = \exp(2\pi it)$, where $i = \sqrt{-1}$. Show that $I(f) = \int_0^1 f(z(t))dt$ is a left invariant integral on U(1).

Problem 3.

Let F be a field, let G be a group and consider the group ring F[G]. Let $i:G\to F[G]$ denote the map that maps $g\in G$ to the element $1g\in F[G]$. Show that this map i has the following universal property: if V is an F-vector space and if $f:G\to \operatorname{GL}(V)\subseteq\operatorname{End}(V)$ is a homomorphism from the group G to the group of invertible linear endomorphisms of V, then there is a unique ring homomorphism $\bar{f}:F[G]\to\operatorname{End}(V)$ such that the diagram



commutes.

A semigroup (S, \cdot) consists of a set S and an associative multiplication \cdot on S. A compact semigroup is a semigroup which carries a compact topology such that \cdot is continuous.

Problem 4.

Let S be a compact semigroup, with $S \neq \emptyset$.

- (1) Show that for every $a \in S$, the set aS is a compact semigroup.
- (2) Show that the set C of all nonempty compact subsemigroups contains minimal elements $T \in S$, i.e. that there is a compact nonempty subsemigroup $T \subseteq S$ which is minimal with this property. [Hint. Use Zorn's Lemma.]
- (3) **Ellis' Theorem.** Show that all elements of such a minimal T are idempotent, i.e. satisfy $t^2 = t$. In particular, conclude that S contains idempotents. [Hint. Consider $A = tT \subseteq T$ and then $B = \{r \in T \mid tr = t\} \subseteq T$.]
- (4) Let G be a Hausdorff topological group, let $K \subseteq G$ be a nonempty compact subset which is a semigroup, i.e. $KK \subseteq K$. Show that K is a subgroup. [Hint. For $s \in K$ consider S = sK and use (3).]

Bonus Problem 1.

The compact interval $[0,1] \subseteq \mathbb{R}$ is under multiplication a compact semigroup, but not a group. Why does this not contradict Problem 8.4(4)?

Bonus Problem 2.

Let G be a group and let p be a prime. Suppose that every element $g \in G$ satisfies $g^p = e$. If p = 2, show that G is abelian. Can one draw the same conclusion if p > 2?