Prof. Dr. Linus Kramer
Raquel Murat García

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Universität Münster

Locally Compact Groups
Sheet 7

Hand in: Friday November 29, 2024 after class in letterbox No 4.

Let X be a nonempty set and let G be a group. A *(left) action* of G on X is a homomorphism $h : G \to \mathrm{Sym}(X)$. For $g \in G$ and $x \in X$ one writes $g(x) = h(g)(x)$ for short. The *stabilizer* of $x \in X$ is the subgroup $G_x = \{g \in G \mid g(x) = x\}$. The *$G$-orbit* of $x \in X$ is the subset $G(x) = \{g(x) \mid g \in G\} \subseteq X$.

Problem 1.
Show that $G_{a(x)} = aG_xa^{-1}$, for all $x \in X$ and $a \in G$. Show that $\bigcap_{x\in X} G_x$ is a normal subgroup of G. (This subgroup is called the *kernel* of the action.)

Problem 2.
Show that $G(y) = G(x)$ holds for all $y \in G(x)$. Conclude that G-orbits are either disjoint or equal. Show also that there is a bijection $G/G_x \to G(x)$ that maps gG_x to $g(x)$.

Problem 3.
Let $H \subseteq G$ be a subgroup and let $x \in X$. Show that $H(x) = G(x)$ holds if and only if $G_xH = G$.

Problem 4.
Let $G = X$ be a group, and put $g(x) = gxg^{-1}$, for $g, x \in G$. Show that this is a left action of G on X and determine the kernel of this action. Conclude that a finite group of prime power order $|G| = p^n$ with $n \geq 1$ has a nontrivial center.
Hint. What are the possible sizes of G-orbits under this action?

Bonus Problem 1.
Let $L \subseteq \mathbb{Z}_p$ be a subgroup of finite index $[\mathbb{Z}_p : L] = m \geq 2$. Show that $m = p^n$ for some $n \geq 1$ and that L is open.