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Universität Münster

Locally Compact Groups
Sheet 6

Hand in: Friday November 22, 2024 after class in letterbox No 4.

Problem 1.
(Poincaré's Lemma) Let G be a group and let $H \subseteq G$ a subgroup of finite index $[G : H] = m$. Show that there is a normal subgroup $N \trianglelefteq G$ of finite index $[G : N] \leq m!$, with $N \subseteq H$.
Hint. Read the proof of Theorem §2.7

Problem 2.
Show that an abstract group G is residually finite if and only if it is isomorphic to a dense subgroup of some profinite group.

A group $G \neq \{e\}$ is called *simple* if it has no normal subgroups besides $\{e\}$ and G.

Problem 3.
Show that a profinite group which is simple is finite. Give an example of a finite simple group and justify your example.

A *simplicial graph* $\Gamma = (V, E)$ consists of a set V of vertices and a set E of two element subsets of V called edges. An *automorphism* of Γ is a permutation of V that maps edges to edges.

Problem 4.
Show that the automorphism group of a simplicial graph Γ is locally compact (in the topology of pointwise convergence) if the graph is connected and if every vertex in contained in finitely many edges.
Hint. Define a relation $R \subseteq V \times V$ such that the automorphisms of Γ are precisely the automorphisms of the relational structure $(V, \{R\})$ and show that the stabilizer of each vertex $v \in V$ is compact, using Problem 5.4. and Bonus Problem 5.1

In these Bonus Problems we do not assume that homomorphisms are continuous.

Bonus Problem 1.
Show that every group homomorphism $\mathbb{Z}_p \to \mathbb{Z}$ is constant.

Bonus Problem 2.
Let G be a profinite group. Show that every group homomorphism $G \to \mathbb{Z}$ is constant.
Hint. Use the previous result and the universal property of $\widehat{\mathbb{Z}}$ to reduce to the case $G = \widehat{\mathbb{Z}}$.