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Locally Compact Groups

Sheet 5

Hand in: Friday November 15, 2024 after class in letterbox No 4.

We consider a set $X \neq \emptyset$ and the symmetric group $\mathrm{Sym}(X)$ consisting of all bijective maps $f : X \to X$. Let $G \subseteq \mathrm{Sym}(X)$ be a subgroup and let $x \in X$. The *G-stabilizer* of x is $G_x = \{g \in G \mid g(x) = x\} \subseteq G$ and the *G-orbit* of x is $G(x) = \{g(x) \mid g \in G\} \subseteq X$.

We view $\mathrm{Sym}(X)$ as a subset of X^X, with the product topology (the topology of pointwise convergence).

Problem 1.
Let $G \subseteq \mathrm{Sym}(X)$ be a subgroup and let $x \in X$. Show that G_x is a subgroup, and that G_x is closed and open in G.

Problem 2.
Show that a subgroup $G \subseteq \mathrm{Sym}(X)$ is closed if and only if some point stabilizer G_x is closed in $\mathrm{Sym}(X)$.

Problem 3.
Let $G \subseteq \mathrm{Sym}(X)$ be a subgroup. Show that the map $G \times X \to X$, $(g, x) \mapsto g(x)$ is continuous. Conclude that G is totally disconnected. Show that every G-orbit is finite if G has compact closure in $\mathrm{Sym}(X)$.

Problem 4.
Let $G \subseteq \mathrm{Sym}(X)$ be a subgroup and suppose that every $x \in X$ has a finite G-orbit. Show that G has compact closure in $\mathrm{Sym}(X)$.

A *relational structure* $M = (X, \mathcal{R})$ consists of a (nonempty) set X, the *universe*, and a (possibly infinite) set $\mathcal{R}$ of subsets $R \subseteq X^n$, for $n \geq 1$, the *n-ary relations*. We let $\mathrm{Sym}(X)$ act on X^n via $g(x_1, \ldots, x_n) = (g(x_1), \ldots, g(x_n))$. The automorphism group $\mathrm{Aut}(M)$ consists of all $g \in \mathrm{Sym}(X)$ with $g(R) = R$ for all $R \in \mathcal{R}$.

Bonus Problem 1.
Show that the automorphism group of a relational structure $M = (X, \mathcal{R})$ is closed in $\mathrm{Sym}(X)$.
Hint. Show that the complement of $\mathrm{Aut}(M)$ *is open.*

Bonus Problem 2.
Prove **Cameron's Theorem**: If $G \subseteq \mathrm{Sym}(X)$ is closed, then there is a relational structure $M = (X, \mathcal{R})$ with $G = \mathrm{Aut}(M)$.
Hint. Consider the set $\mathcal{R}$ of all G-orbits $R = G(x_1, \ldots, x_n) \subseteq X^n$, for $n \geq 1$.