Prof. Dr. Linus Kramer
Raquel Murat García

Winter term 2024/25
Universität Münster

Locally Compact Groups

Sheet 4

Hand in: Friday November 8, 2024 after class in letterbox No 4.

Problem 1.
Let X be a Baire space and let $Y \subseteq X$ be a dense subset which is a G_δ-set, i.e. $Y = \bigcap_{k=0}^{\infty} U_k$, each U_k open. Show that Y is a Baire space. Conclude that the set of irrational numbers $\mathbb{R} - \mathbb{Q}$ is a Baire space.
Hint. Suppose that $B_j \subseteq Y$ are subsets which are open and dense in Y. To show that $\bigcap_{j=0}^{\infty} B_j \subseteq Y$ is dense in Y, write $B_j = Y \cap W_j$ with $W_j \subseteq X$open and consider the sets $U_k \cap W_j$.

Problem 2.
A *topological vector space* V is a real vector space which carries a Hausdorff topology such that $(V, +)$ is a topological group, and such that scalar multiplication $\mathbb{R} \times V \to V$ is continuous.
Let V, W be topological vector spaces and let $f : V \to W$ be a morphism of topological groups. Show that f is an $\mathbb{R}$-linear map.
Hint. Show first that f is $\mathbb{Q}$-linear.

Problem 3.
Let $F = \mathbb{F}_p$ denote the finite field with p elements, for a prime p. Consider the vector space $V = F^{\mathbb{N}}$ and its linear subspace $V_0 = \bigoplus_{k=0}^{\infty} F$ consisting of all sequences which are eventually 0. Show that the linear map $f_0 : V_0 \to F$ that maps $(c_k)_{k \in \mathbb{N}}$ to $\sum_k c_k$ extends to a linear map $f : V \to F$. Show that f cannot be continuous (with respect to the product topology on V and the discrete topology on F). Conclude that $\ker(f)$ is not Baire measurable.

Problem 4.
Let G be a locally compact metrizable group. Show that G is Weil complete.

Bonus Problem 1.
Let G be a locally compact group and let $H \subseteq G$ be a closed subgroup. Show that G/H is locally compact.