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Universität Münster

Locally Compact Groups

Sheet 3

Hand in: Thursday October 31, 2024 before class in letterbox No 4.

Problem 1.
Let $f : G \to K$ be a morphism of topological groups and let $N \trianglelefteq G$ be a normal subgroup which is contained in $\ker(f)$. Show that f factors uniquely as

$$\begin{array}{ccc} G & \xrightarrow{\;f\;} & K \\ {\scriptstyle q}\big\downarrow & \nearrow_{\bar{f}} & \\ G/N & & \end{array}$$

with a morphism $\bar{f}$, where $q(g) = gN$. Show that $\bar{f}$ is open if and only if f is open.
Hint. You may use the homomorphism theorem from algebra. What do you need to prove?

Problem 2.
Let G be a Hausdorff topological group and let $K \subseteq G$ be a compact subgroup. Show that the map $q : G \to G/K$ is closed.

Problem 3.
A sequence $(g_k)_{k\in\mathbb{N}}$ in a Hausdorff topological group G is called a *right Cauchy sequence* if for every identity neighborhood $V \subseteq G$ there is an $m \in \mathbb{N}$ such that $g_j g_k^{-1} \in V$ holds for all $j, k \geq m$. Show that a metrizable topological group is Weil complete if and only if every right Cauchy sequence converges.

Problem 4.
Let I be an infinite set with the discrete topology. Consider the monoid $S = I^I$ consisting of all maps $f : I \to I$, with the product topology. (This is the topology of pointwise convergence.) Let $\mathrm{Sym}(I) \subseteq S$ denote the group of all bijective maps.
(i) Show that multiplication $(f, h) \longmapsto f \circ h$ is continuous on S.
(ii) Show that inversion is continuous on $\mathrm{Sym}(I)$ and that $\mathrm{Sym}(I)$ is therefore a topological group.
Hint. For $f \in I^I$ a neighborhood basis of f in S consists of the sets $N(f, E) = \{h \in I^I \mid f|_E = h|_E\}$, where E varies over all finite subsets of I.

Bonus Problem 1.
(Notation as in Problem 3.4) Show that the closure of $\mathrm{Sym}(I)$ in S is the set of all injective maps $f : I \to I$. Conclude that $\mathrm{Sym}(\mathbb{N})$ is not Weil complete.
Hint: Show first that the set of non-injective maps $f : I \to I$ is open.

Note: $\mathrm{Sym}(\mathbb{N})$ is what is called a *Polish group*: a second countable completely metrizable topological group.