Prof. Dr. Linus Kramer
Raquel Murat García

Winter term 2024/25
Universität Münster

Locally Compact Groups

Sheet 2

Hand in: Friday October 25, 2024 before class in letterbox No 4.

Problem 1.
The set $H = \{2^k \mid k \in \mathbb{Z}\}$ is a discrete subgroup of the multiplicative group of the reals which is not closed in $\mathbb{R}$. Why does this not contradict §1.9 Cor. B?

Problem 2.
Let $H \subseteq \mathbb{R}$ be a discrete subgroup. Show that $H = r\mathbb{Z}$ for a unique real $r \geq 0$.
Hint. If $H \neq \{0\}$ *put* $r = \inf\{h \in H \mid h > 0\}$ *and show that* $r \in H$.

Problem 3.
Let $H \subseteq \mathbb{R}$ be a non-discrete subgroup. Show that H is dense in $\mathbb{R}$.
Conclude that $\mathbb{Z} + r\mathbb{Z} \subseteq \mathbb{R}$ is dense if r is not rational.
Hint. Show that for every integer $m \geq 1$ *here is some* $h_m \in H$ *with* $0 < h_m \leq \frac{1}{m}$. *Then show that for every real number* r *and every* $\varepsilon > 0$ *there are integers* k, ℓ *with* $|r - kh_\ell| \leq \varepsilon$.

Problem 4.
Let G be a topological group and let $\mathcal{N}$ be a neighborhood basis of the identity element e. Let $X \subseteq G$ be a subset. Show that $\overline{X} = \bigcap\{VX \mid V \in \mathcal{N}\}$.

Bonus Problem 1.
Show that every morphism $f : \mathrm{U}(1) \to \mathrm{U}(1)$ is of the form $f(z) = z^k$, for some unique $k \in \mathbb{Z}$.
Hint. Consider the morphism $p : \mathbb{R} \to \mathrm{U}(1)$, $p(t) = \exp(it)$ *and use Exercise 1.3*

Bonus Problem 2.
Let $H \subseteq \mathbb{R}$ be a connected nontrivial subgroup. Show that $H = \mathbb{R}$.