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Universität Münster

Locally Compact Groups
Sheet 1

Hand in: Friday October 18, 2024 before class in letterbox No 4.

Problem 1.
Let G, K be topological groups and let $f : G \to K$ be a group homomorphism. Show that f is continuous if it is continuous at some point $a \in G$.

Problem 2.
We consider the additive group $(\mathbb{R}, +)$ with the usual topology. Then $\mathbb{R}$ is a topological group. Let $f : \mathbb{R} \to \mathbb{R}$ be a morphism. Show that there is a unique real number r such that $f(x) = rx$ holds for all $x \in \mathbb{R}$.
Hint. Show first that there is a unique $r \in \mathbb{R}$ such that $f(x) = rx$ holds for all $x \in \mathbb{Q}$.

Problem 3.
Let $h : \mathbb{R} \to \mathrm{U}(1)$ be a morphism of topological groups, where $\mathrm{U}(1) = \{\exp(it) \mid t \in \mathbb{R}\} \subseteq \mathbb{C}$ is the circle group, and $i = \sqrt{-1}$. Show that there is a real number r such that $h(x) = \exp(irx)$ holds for all $x \in \mathbb{R}$.
Hint. Consider the universal covering $p : \mathbb{R} \to \mathrm{U}(1)$, with $p(t) = \exp(it)$. Show then that the unique lift $\tilde{h}$ of h with initial condition $\tilde{h}(0) = 0$ is a morphism. Use for this that p is a morphism.
If you are not familiar with the complex exponential function you can put $\mathbb{C} = \mathbb{R}^2$, $\mathrm{U}(1) = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ and $\exp(it) = (\cos(t), \sin(t))$ in cartesian coordinates.

Problem 4.
Let G be a topological group, let $a \in G$ and let $U \subseteq G$ be a neighborhood of a.
(a) Show that there is an open neighborhood V of e such that $VVa \subseteq U$.
(b) Show next that Va and VVa are open neighborhoods of a.
(c) Show next that $\overline{Va} \subseteq VVa$.
(d) Conclude that G is a regular space (T_3) if it is a T_1-space.

Bonus Problem 1.
No topological group is homeomorphic to the unit interval $[0, 1]$.