§4 Duality

Recall the following from linear algebra. Let V be a vector space (over a field k). The dual space is $V^* = \mathrm{Hom}(V, k)$. If $(b_i)_{i \in I}$ is a basis for V, then $V \cong \bigoplus_{i \in I} k$ (direct sum) which $V^* \cong \prod_{i \in I} k$ (direct product).

Hence $\dim(V) = \dim(V^*)$ if V has finite dimension, while $\dim(V) < \dim(V^*)$ if V has infinite dimension. (1)

There is a canonical injective map $V \to V^{**}$, $u \longmapsto [\alpha \mapsto \alpha(u)]$ $\quad u \in V, \alpha \in V^*$

Hence $V \to V^{**}$ is a canonical isomorphism if $\dim(V) < \infty$.

We want to generalize this to (locally) compact abelian groups.

1. Def Let $(A, +)$, $(D, +)$ be abelian groups. Then $\mathrm{Hom}(A, D)$ is also an abelian group if we put $(\alpha + \beta)(x) = \alpha(x) + \beta(x)$.

(1)

Note also: a linear map $F: V \to W$

induces a linear map $V^* \xleftarrow{F^*} W^*$

via $F^*(\beta) = \beta \circ F$

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ {\scriptstyle F^*\beta}\downarrow & & \downarrow{\scriptstyle \beta} \\ k & = & k \end{array}$$

Indeed, we have for $x, y \in A$

$$(\alpha+\beta)(x+y) = \alpha(x+y) + \beta(x+y) = \alpha(x)+\alpha(y)+\beta(x)+\beta(y)$$

$$\overset{\uparrow}{=} \alpha(x)+\beta(x)+\alpha(y)+\beta(y) = (\alpha+\beta)(x) + (\alpha+\beta)(y). \quad \square$$

D abelian

If $A \xrightarrow{f} B$ is a homomorphism of abelian groups, we obtain a homomorphism $\mathrm{Hom}(A,D) \xleftarrow{f^*} \mathrm{Hom}(B,D)$

via $f^*(\beta) = \beta \circ f$, for $\beta \in \mathrm{Hom}(B,D)$

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle f^*\beta = \beta\circ f}\downarrow & & \downarrow{\scriptstyle \beta} \\ D & = & D \end{array}$$

$$f^*(\beta_1+\beta_2) = (\beta_1+\beta_2)\circ f = \beta_1\circ f + \beta_2 \circ f = f^*\beta_1 + f^*\beta_2.$$

Example There is an isomorphism $\mathrm{Hom}(\mathbb{Z}, D) \xrightarrow[\cong]{h} D$

via $h(\xi) = \xi(1)$, $\xi : \mathbb{Z} \to D$.

Indeed, $h(\xi+\xi) = (\xi+\xi)(1) = \xi(1)+\xi(1) = h(\xi)+h(\xi)$

h is injective: $h(\xi) = 0 \Rightarrow \xi(1) = 0 \Rightarrow \xi(m) = 0$

for all $m \Rightarrow \xi = 0 \Rightarrow \ker(h) = \{0\}$

h is surjective: Let $d \in D$, put $\xi(m) = m \cdot d$

$\Rightarrow h(\xi) = d$. $\square$

2. Def Let $A \xrightarrow{f} B \xrightarrow{h} C$ be homomorphisms of abelian grps. We call this exact at B if $f(A) = \ker(h)$, i.e. if

$$h(b) = 0 \iff b = f(a) \text{ for some } a \in A.$$

A sequence of homomorphism of abelian grps

$$\to A_k \xrightarrow{f_k} A_{k+1} \xrightarrow{f_{k+1}} A_{k+2} \to \cdots$$

is called an exact sequence if it is exact at every A_ℓ where there is an ingoing and an outgoing homomorphism at A_ℓ.

Examples (a) $0 \to A \xrightarrow{i} B$ is exact $\iff$ $\ker(i) = 0 \iff i$ is injective

(b) $B \xrightarrow{q} C \to 0$ is exact $\iff q(B) = C$

$\iff$ q is surjective

(c) $0 \to A \xrightarrow{i} B \xrightarrow{q} C \to 0$ is exact

$\iff$ i is injective, q is surjective, $i(A) = \ker(q)$

This is called a short exact sequence.

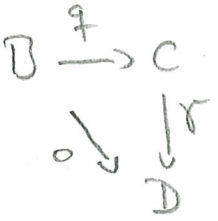
#

3. **Lemma** Let A, B, C, D be abelian groups.

If $A \xrightarrow{i} B \xrightarrow{q} C \to 0$ is an exact sequence, then $\mathrm{Hom}(A,D) \xleftarrow{i^*} \mathrm{Hom}(B,D) \xleftarrow{q^*} \mathrm{Hom}(C,D) \leftarrow 0$ is also an exact sequence.

proof. Since $q \circ i = 0$, $(q \circ i)^* = i^* \circ q^* = 0$,

Claim q^* is injective



Suppose $q^*(r) = 0$, i.e. $r \circ q = 0$

Then $r(q(B)) = r(C) = \{0\} \Rightarrow r = 0$.

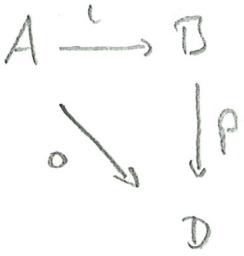
$\uparrow$ q surjective

□

We know already that $q^*(\mathrm{Hom}(C,D)) \subseteq \ker(i^*)$

and we need to show "$=$".

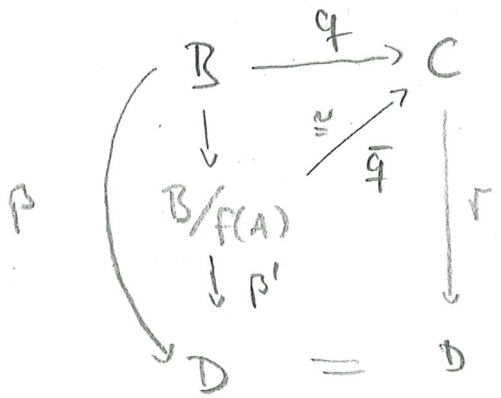
Let $\beta \in \ker(i^*)$, i.e. $\beta \circ i = 0$



By the homomorphism theorem, we have

an isomorphism $B/_{i(A)} \xrightarrow[\cong]{\bar{q}} C$

and a homomorphism $\bar{\beta}: B/_{i(A)} \to D$, whence



i.e. $\beta = \underbrace{\beta' \circ (\bar{q})^{-1}}_{= r} \circ q$

$= q^* r$

□

Corollary Let $m \geq 0$. Then

$$\mathrm{Hom}(\mathbb{Z}/m, D) \cong \underbrace{\ker(d \mapsto md)}_{=\{d \in D \mid md = 0\}} \subseteq D$$

proof Consider $\mathbb{Z} \xrightarrow{\cdot m} \mathbb{Z} \to \mathbb{Z}/m \to 0$

and dualize $\underbrace{\mathrm{Hom}(\mathbb{Z}, D)}_{\cong D} \xleftarrow{\cdot m} \underbrace{\mathrm{Hom}(\mathbb{Z}, D)}_{\cong D} \leftarrow \mathrm{Hom}(\mathbb{Z}/m, D) \leftarrow 0$ □

4. Let $(A_i)_{i \in I}$ be a family of abelian grps. The product $A = \prod_{i \in I} A_i$ has a **universal property**: given an abelian grp B and homomorphisms $\beta_i : B \to A_i$, there is a unique homomorphism $\beta : B \to A$ such that $\beta_i = \mathrm{pr}_i \circ \beta$, namely

$$\beta(b) = (\beta_i(b))_{i \in I} \in A.$$

There is a dual notion, the **coproduct** or **direct sum**

$$\bigoplus_{i \in I} A_i = \{ (a_i)_{i \in I} \mid a_i \in A_i, \text{ almost all } a_i = 0 \}$$

One writes the elements of $\bigoplus_{i \in I} A_i$ as **formal sums** $\sum_{i \in I} a_i$ (where only **finitely many** $a_i \neq 0$).

For each i we have an **inclusion map** $h_j : A_j \to \bigoplus_{i \in I} A_i$

$$(h_j(a))_i = \begin{cases} 0 & \text{if } i \neq j \\ a & \text{if } i = j \end{cases}$$

The universal property is as follows. Given an abelian grp G and homomorphism $\gamma_i : A_i \to G$, there is a unique homomorphism $\gamma : \bigoplus_{i \in I} A_i \to G$ with

$\gamma_i = \gamma \circ k_i$, namely $\gamma\left(\sum_{i \in I} a_i\right) = \underbrace{\sum_{i \in I} \gamma_i(a_i)}_{\text{finite sum!}}$

Note: if I is finite, then $\bigoplus_{i \in I} A_i = \prod_{i \in I} A_i$.

Lemma We have $\mathrm{Hom}\left(\bigoplus_{i \in I} A_i, D\right) \cong \prod_{i \in I} \mathrm{Hom}(A_i, D)$

and $\mathrm{Hom}\left(D, \prod_{i \in I} A_i\right) \cong \prod_{i \in I} \mathrm{Hom}(D, A_i)$

proof Given $f : \bigoplus_{i \in I} A_i \to D$, put $f_i = f \circ k_i$, where $k_i : A_i \to \bigoplus_{k \in I} A_k$ as before. The first isomorphism maps F to $(f_i)_{i \in I}$.

Given $\varphi : D \to \prod_{i \in I} A_i$, put $\varphi_i = pr_i \circ \varphi$. The second isomorphism maps φ to $(\varphi_i)_{i \in I}$.

(The rest is easy / exercise). □

5. Def Let A be a topological abelian group.

The character group is $X(A) = \mathrm{Mor}(A, \mathbb{T})$,

where $\mathbb{T} = U(1) = \{z \in \mathbb{C} \mid |z| = 1\}$ is the circle group. Thus $X(A)$ is an abelian group, with multiplication $(\alpha \cdot \beta)(a) = \alpha(a) \cdot \beta(a)$, $\alpha, \beta : A \to \mathbb{T}$

The neutral element is $1 : a \mapsto 1 \in \mathbb{T}$, the inverse of α is $\bar{\alpha}$, since $\alpha\bar{\alpha} = 1$.

Examples

(a) $A = \mathbb{Z}$ (discrete topology)

$\rightsquigarrow X(A) = \mathrm{Mor}(A, \mathbb{T}) = \mathrm{Hom}(A, \mathbb{T}) \cong \mathbb{T}$

(↑ discont)

(b) $A = \mathbb{T}$, then $X(A) = \mathrm{Mor}(\mathbb{T}, \mathbb{T}) \cong \mathbb{Z}$

cp. Bonus Problem 2.1: every morphism $\alpha : \mathbb{T} \to \mathbb{T}$ is of the form $\alpha(z) = z^k$ for some $k \in \mathbb{Z}$

(c) $A = \mathbb{Z}/m$, $X(A) = \mathrm{Mor}(A, \mathbb{T}) = \mathrm{Hom}(A, \mathbb{T})$

$= \{z \in \mathbb{T} \mid z^m = 1\}$, cp. §4.3

$\cong \mathbb{Z}/m$

In these examples we have $X(X(A)) \cong A$! (*)

This leads to Pontrjagin Duality

[L. Pontrjagin, Sowjet. Math, 1908–1988]

(*) This is checking – we need a topology on $X(A)$.

6. The structure of finitely generated abelian groups

(after J.S. Milne)

Let $(A,+)$ be an abelian grp. We call a generating set $x_1,\dots,x_k$ a <u>basis for A</u> if

$$\sum_{i=1}^{k} m_i a_i = 0 \quad \Rightarrow \quad m_1 a_1 = m_2 a_2 = \dots = m_k a_k = 0$$

for all integers $m_1,\dots,m_k$.

If $\{x_1,\dots,x_k\}$ is a basis, then $A \cong \langle x_1\rangle \oplus \dots \oplus \langle x_k\rangle$

$(\langle x\rangle = \{ mx \mid m\in\mathbb{Z}\})$

<u>Lemma</u> Suppose that $x_1,\dots,x_k$ generate A.

If $c_1,\dots,c_k \in \mathbb{N}$ with $\gcd(c_1,\dots,c_k)=1$, then

there are generators $y_1,\dots,y_k$ with $y_1 = c_1x_1 + \dots + c_kx_k$.

<u>proof</u> By induction on $s = c_1+\dots+c_k > 0$.

Clear if $s=1$. (why?)

Assume $s\geq 2$. wlog $c_1 \geq c_2 > 0$, and we have the following: (a) $\{x_1, x_1+x_2, x_3,\dots,x_k\}$ generate A

(b) $\gcd(c_1-c_2, c_2,\dots,c_k) = 1$

(c) $(c_1-c_2)+c_2+\dots+c_k < s$.

Hence there are generators $y_1,\dots,y_k$ with

$$y_1 = (c_1-c_2)x_1 + c_2(x_1+x_2) + \dots + c_kx_k$$
$$= c_1x_1 + c_2x_2 + \dots + c_kx_k. \qquad \square$$

Theorem Every finitely generated abelian group $(A,+)$ has a basis. Hence $A \cong \mathbb{Z}^l \oplus \mathbb{Z}/n_1 \oplus \cdots \oplus \mathbb{Z}/n_r$ for $l \geq 0$, $r \geq 0$, $n_1, \ldots, n_r \geq 2$

proof Let k denote the minimal number of generators and let $x_1, \ldots, x_k$ be a generating set such that the order t of x_1 is as small as possible.

Claim: $A \cong \langle x_1 \rangle \oplus \underbrace{\langle x_2, \ldots, x_k \rangle}_{\text{generated by } \{x_2, \ldots, x_k\}}$

pf of claim If this is false, we have a relation

$$m_1 x_1 + \cdots + m_k x_k = 0, \quad \text{with } m_1 x_1 \neq 0$$

wlog $m_i \geq 0$ (replace x_i by $-x_i$) and $0 \leq m_1 < t$.

Put $d = \gcd(m_1, \ldots, m_n) > 0$ and $c_i = \frac{m_i}{d}$.

By the lemma, there is a generating set $y_1, \ldots, y_k$ with $y_1 = c_1 x_1 + \cdots + c_k x_k$. But $d \cdot y_1 = 0$

$d \leq m_1 < t$ ↯ Hence the claim is true. □

If $x_2', \ldots, x_k'$ is a basis for $\langle x_2, \ldots, x_k \rangle$ (by induction), then $x_1, x_2', \ldots, x_k'$ is a basis for A.

□

7. Lemma Let $(A,+)$ be an abelian grp. Then

$X(A) = \mathrm{Hom}(A, \mathbb{T})$ is a cmpact abln grp with respect to the subspace topology, $X(A) \subseteq \mathbb{T}^A$

For $A = \mathbb{Z}$, we have an isomphism of cmpact grps

$X(\mathbb{Z}) \cong \mathbb{T}$ via $\alpha \mapsto \alpha(1)$

Proof Let $\alpha : A \to \mathbb{T}$ be a map. Then $\alpha \in X(A)$ holds if $\alpha(u+v)\bar{\alpha}(u)\bar{\alpha}(v) = 1$ for all $u,v \in A$,

hence $X(A) = \bigcap_{u,v \in A} \{ \alpha \in \mathbb{T}^A \mid \alpha(u+v)\bar{\alpha}(u)\bar{\alpha}(v) = 1 \}$

is a closed subgrp of the cmpact grp $\mathbb{T}^A$.

For $A = \mathbb{Z}$, the map $\alpha \mapsto \alpha(1)$, $X(\mathbb{Z}) \to \mathbb{T}$ is cntins and bijective, hence a homeomphism.

8. Lemma Let $(A,+), (B,+)$ be abelian groups. Then

$h : X(A) \times X(B) \xrightarrow{\cong} X(A \times B)$ is an isomphism of topological groups, $h(\alpha,\beta)(u,v) = \alpha(u)\beta(v)$

proof The map h is continuous and injective.

The map $X(A \times B) \to X(A)$, $\gamma \mapsto (u \mapsto \gamma(u,0)) = \gamma_A$

is also cntins, similarly $X(A \times B) \to X(B)$, $\gamma \mapsto (v \mapsto \gamma(0,v)) = \gamma_B$

and $\gamma_A \cdot \gamma_B = \gamma$

$\Rightarrow$ cts map $X(A \times B) \to X(A) \times X(B)$ inverse to h.

□

Cor Let F be a finite abelian group.

Then $X(\mathbb{Z}^\ell \times F) \cong \mathbb{T}^\ell \times X(F) \cong \mathbb{T}^\ell \times F$ (as a compact grp)

and $X(\mathbb{T}^\ell \times F) \cong \mathbb{Z}^\ell \times X(F) \cong \mathbb{Z}^\ell \times F$ (as an abelian grp)

proof Follows from §4.8 and §4.5. □

9. Def Let $(A,+)$ be an abelian grp / compact abelian grp.

We define $i_A : A \to XX(A)$ via $i_A(a)(\alpha) = \alpha(a)$,

for $\alpha \in X(A)$.

Note: (a) if A is discrete, then $i_A(a)$ is a morphism, $i_A(a) \in XX(A)$.

(b) If A is compact, then $a \longmapsto i_A(a)(\alpha) = \alpha(a)$ is continuous for each α, hence i_A is a morphism.

(c) If $i_A(a) = 1$, then $\alpha(a) = 1$ for all $\alpha \in X(A)$

If A is compact, this implies that $a = 0$ by §3.26. Hence i_A is injective.

If A is discrete, this implies also that $a = 0$ by the following result.

10. Theorem (R. Baer)

[R. Baer, 1902-1979, German mathematician]

Let $(A,+)$ be an abelian grp. Let $H \leq A$ be a subgrp, let $\alpha : H \to \mathbb{T}$ be a homomorphism

Then there is a homomorphism $\tilde{\alpha} : A \to \mathbb{T}$ with

$\tilde{\alpha}|_H = \alpha$.

i.e. the map $X(H) \leftarrow X(A)$ (from $H \hookrightarrow A$)

is surjective.

Proof Let P denote the set of all pairs $H' \subseteq A$,

$\alpha' : H' \to \Pi$ homomorphic, H' subgp, $\alpha'|_H = \alpha$.

$$\begin{array}{ccc} H & \xrightarrow{\alpha} & \Pi \\ \downarrow & \nearrow_{\alpha'} & \\ H' & & \end{array}$$

Thus $P \neq \phi$, and $(P, \leq)$ is a poset if we put

$(\alpha', H') \leq (\alpha'', H'') \overset{\text{Def}}{\iff} H' \subseteq H''$ and $\alpha''|_{H'} = \alpha'$.

Every linearly ordered subset $P_0 \subseteq P$ has an upper bound (the union), hence P contains maximal elements.

Let $(\alpha', H') \in P$ be maximal. If $H' \neq A$, choose $a \in A - H'$ and extend α' to α'' on $H' = \langle a \rangle + H'$ as follows: (a) if $\langle a \rangle \cap H' = \{0\}$, put $\alpha''(a) = 1$

(b) if $\langle a \rangle \cap H' \neq \{0\}$, choose $m > 1$ minimal with $ma \in H'$, choose $z \in \Pi$ with $z^m = \alpha'(ma)$ and put $\alpha''(a) = z$.

This extends α' to α''.

But (α', H') was maximal, whence $H' = A$. □

This implies that $X(H) \leftarrow X(A)$ is surjective.

Hence $i_A : A \to XX(A)$ is injective.

11. Lemma The map i_A is an isomorphism for $A = \mathbb{Z}, \mathbb{T}$ or if A is finite abelian.

proof $A = \mathbb{Z}$: $i_{\mathbb{Z}}(1)(\alpha) = \alpha(1)$, hence

$i_{\mathbb{Z}}(1) = h : X(\mathbb{Z}) \xrightarrow{\cong} \mathbb{T}$ as in §4.7.

Moreover, $X(\mathbb{T}) \cong \mathbb{Z}$ via $(z \longmapsto z^k) \longmapsto k$,

cp. §4.5 Example (b). Thus $i_{\mathbb{Z}}(1) = 1$

implies the isomorphism $XX(\mathbb{Z}) \cong X(\mathbb{T})$. □

$A = \mathbb{T}$: $i_{\mathbb{T}}(a)(1) = a$ and $X(\mathbb{T}) \cong \mathbb{Z}$,

$XX(\mathbb{T}) \cong X(\mathbb{Z})$. □

A finite : i_A is injective and $X(A) \cong A$ by

Example §4.5 (c) and §4.6. □

Corollary i_A is an isomorphism if $A \cong \mathbb{Z}^l \times F$

or if $A \cong \mathbb{T}^l \times F$, F finite abelian.

12. Proposition Let $(A, +)$ be an abelian grp, let $B \subseteq X(A)$ be a closed subgroup. If B separates points i.e. if $\bigcap_{\beta \in B} \ker(\beta) = \{0\}$,

then $B = X(A)$.

proof ① Suppose that A is finitely generated.

Assume towards a contradiction that $B \neq X(A)$.

Hence there is a non-constant character $\alpha \in X(X(A)/B)$

by Peter-Weyl §3.2.1. Put $q: X(A) \to X(A)/B$.

Then $q^*(\alpha)$ is constant on B, hence $XX(A) \xrightarrow{j^*} X(B)$

is not injective ($j: B \hookrightarrow X(A)$), hence $A \xrightarrow{i_A} XX(A) \xrightarrow{j^*} X(B)$

is not injective, hence there is $a \in A$ such that $a \neq 0$

but $\beta(a) = 1$ for all $\beta \in B$ ↯

② The general case Let $\alpha \in X(A)$, let $a_1, \dots, a_m \in A$,

let $U_j \subseteq \mathbb{T}$ be open with $a_j \in U_j$, $j = 1, \dots, m$.

Then $W = \{ \xi \in X(A) \mid \xi(a_j) \in U_j, j = 1, \dots, m \}$

is a typical neighborhood of α in $X(A)$.

Put $A' = \langle a_1, \dots, a_m \rangle$ and

Put $B' = \{ \beta|_{A'} \mid \beta \in B \}$. Then B' separates

points in A', hence $B' = X(A')$ by ①. Hence

there is $\beta \in B$ such that $\beta(a_j) = \alpha(a_j)$ for $j = 1, \dots, m$.

In particular, $W \cap B \neq \emptyset$. Thus B is dense in $X(A)$.

But B is also closed, hence $B = X(A)$. □

13. Theorem (Pontryagin Duality for compact abelian grps)

(i) If A is a compact abelian group, then $i_A : A \to XX(A)$ is an isomorphism of compact groups

(ii) If A is an abelian group, then $i_A : A \to XX(A)$ is a group isomorphism.

proof (i): Let $(A,+)$ be compact abelian. Then i_A is injective by §4.9. Since $B = i_A(A) \leq XX(A)$ is compact and separates points in $X(A)$, i_A is surjective by §4.12. Since A is compact and i_A is continuous, i_A is an isomorphism.

(ii): Let $(A,+)$ be abelian. By Baer's Theorem §4.10, i_A is injective. Put $B = i_A(A)$. If $B \neq XX(A)$, then there is a nontrivial character $\xi : XX(A)/B \to \mathbb{T}$ by Baer's Theorem. Consider $q : XX(A) \to XX(A)/B$. Then $q^*\xi \neq 1$, $q^*\xi \in XXX(A)$, $q^*\xi|_B = 1$. Thus $q^*\xi$ is not in the image of $i_{X(A)} : X(A) \to XXX(A)$, contradicting (i). $\lightning$ □

Remark If $A \xrightarrow{f} B$ is a homomorphism of abelian groups (or a morphism of topol. gps), then the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle i_A}\downarrow\cong & & \downarrow{\scriptstyle i_B} \\ XX(A) & \xrightarrow{f^{**}} & XX(B) \end{array}$$

commutes.

Thus X gives us an **equivalence of categories** between the category $\underline{Ab} = \{$ homomorphisms between abelian groups $\}$ and $\underline{CAb} = \{$ morphisms between topol. abelian gps $\}$

14. **Def** Let $f: A \to B$ be a homomorphism in $\underline{Ab}$ or in $\underline{CAb}$. We call f

epic if the following holds:
if $A \xrightarrow{f} B \underset{h_2}{\overset{h_1}{\rightrightarrows}} C$ commutes, i.e. $h_1 \circ f = h_2 \circ f$,
then $h_1 = h_2$ (for h_1, h_2 in $\underline{Ab}$ resp. $\underline{CAb}$)

monic if the following holds:
if $D \underset{j_2}{\overset{j_1}{\rightrightarrows}} A \xrightarrow{f} B$ commutes, i.e. $f \circ j_1 = f \circ j_2$,
then $j_1 = j_2$ (for j_1, j_2 in $\underline{Ab}$ resp $\underline{CAb}$)

Lemma f is epic $\Leftrightarrow$ f is surjective (in $\underline{Ab}$ or $\underline{CAb}$)
f is monic $\Leftrightarrow$ f is injective (in $\underline{Ab}$ or $\underline{CAb}$)

Proof f surjective $\Rightarrow$ f epic (✓)

f injective $\Rightarrow$ f monic (✓)

Assum that f is epic and consid

$$A \xrightarrow{f} B \underset{0}{\overset{q}{\rightrightarrows}} B/f(A) \quad \Rightarrow \quad q = 0 \Rightarrow f(A) = B.$$

Assum f is monic and consid

$$\ker(f) \underset{0}{\overset{i}{\rightrightarrows}} A \xrightarrow{f} B \quad \Rightarrow \quad i = 0 \Rightarrow \ker(f) = \{0\}.$$

□

Corollary If f is a homomorphism in $\underline{Ab}$ resp. $\underline{CAb}$,

then: f is monic $\Leftrightarrow$ f^* is epic

f is epic $\Leftrightarrow$ f^* is monic

In particular: if $0 \to A \to B \to C \to 0$ is a short exact sequence, then $0 \leftarrow X(A) \leftarrow X(B) \leftarrow X(C) \leftarrow 0$ is a short exact sequence (in $\underline{Ab}$ resp $\underline{CAb}$).

proof Suppose $A \xrightarrow{f} B$ is monic. Consid

$$D \underset{h_2}{\overset{h_1}{\leftleftarrows}} X(A) \xleftarrow{f^*} X(B)$$

$$X(D) \underset{h_2^*}{\overset{h_1^*}{\rightrightarrows}} XX(A) \xrightarrow{f^{**}} XX(B) \quad \Rightarrow \quad h_1^* = h_2^*$$

(f^{**} monic)

$$\Rightarrow h_1^{**} = h_2^{**}$$

$$\Rightarrow h_1 = h_2$$

□

the rest is similar.

15. Def For an abelian group A put $\mathrm{Tor}(A) = \{a \in A \mid a \text{ has finite order}\}$. Since A is abelian, this is a subgroup of A, the torsion subgroup.

Lemma Let $(A,+)$ be an abelian grp, let $H \subseteq X(A)$ be a connected subgrp. Then H annihilates $\mathrm{Tor}(A)$, i.e. $\eta(a) = 1$ for all $\eta \in H$, $a \in \mathrm{Tor}(A)$.

proof Consider the monic $\langle a \rangle \xrightarrow{i} A$ for $a \in \mathrm{Tor}(A)$

$\Rightarrow$ $X(\langle a \rangle) \xleftarrow{i^*} X(A)$ epic $\Rightarrow$ $i^*(H) = \{1\}$. $\square$

($X(\langle a \rangle)$ finite)

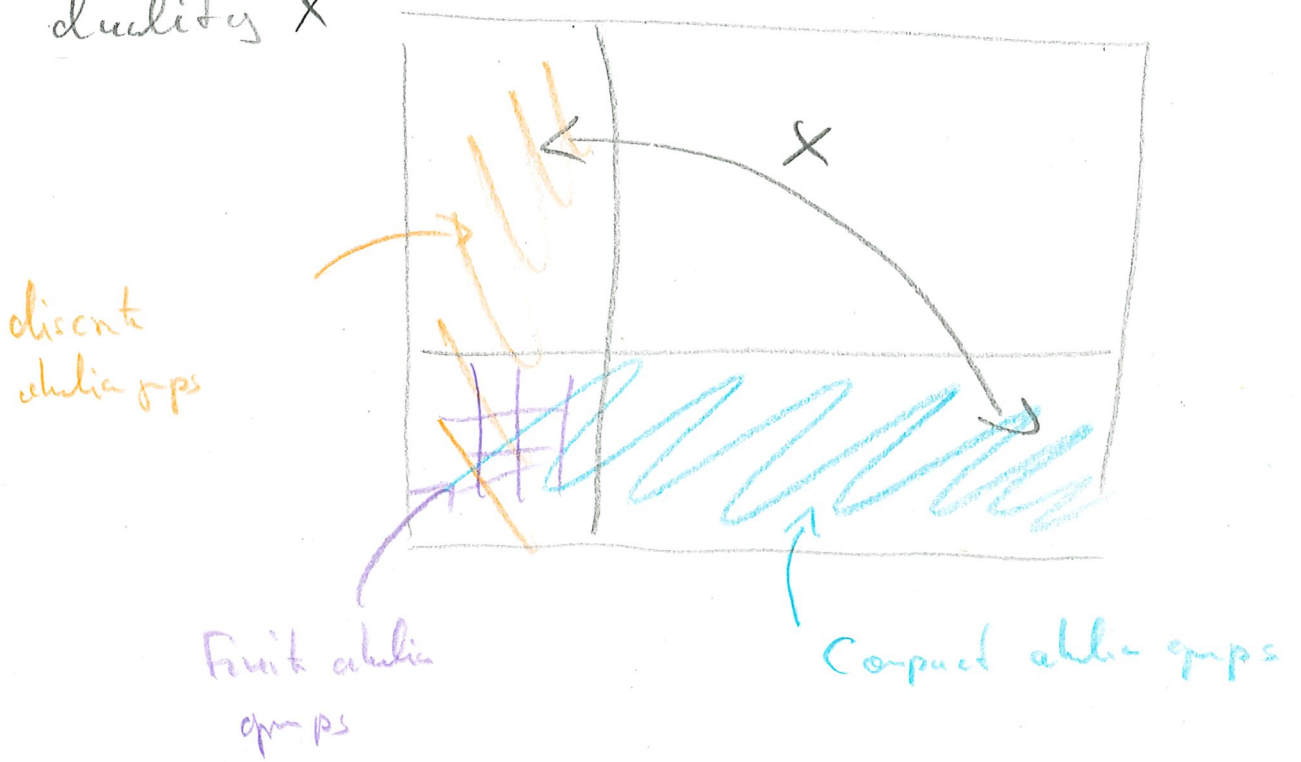
Prop Let A be a cmpctd abelian grp. Then A is connected if and only if $\mathrm{Tor}(X(A)) = \{1\}$.

proof If A is connd & $F \subseteq X(A)$ is a finite subgrp, then $F = \{e\}$ by the Lemma above.

If A is not connd, then the profinite grp A/A° has a proper open subgrp (by van Dantzig), hence H is an open subgrp $H \subseteq A$ with $H \neq A$ $\Rightarrow$ consider the epic $A \xrightarrow{q} \underbrace{A/H}_{\text{finite}}$

$X(A) \xleftarrow[\text{monic}]{} X(A/H) \Rightarrow \mathrm{Tor}(X(A)) \neq \{e\}$ $\square$

Out look Pontryagin Duality gives as a duality X



Pontryagin - von Kampen Duality completes this to a duality