§3 Hilbert modules and the Peter-Weyl Theorem

Recall from §2.8 that a topological group G is profinite if and only if it is isomorphic to a closed subgrp of a product $\prod_{i\in I} F_i$ of finite groups F_i.

Our aim is to prove the following: a topological group G is compact if and only if it is isomorphic to a closed subgrp of a product $\prod_{i\in I} K_i$, where each $K_i \leq GL_{m_i}(\mathbb{C})$ is a compact matrix group. #

1. Def Let X be a Hausdorff space. We put

$$C_c(X,\mathbb{C}) = \{ f : X \to \mathbb{C} \mid f \text{ continuous with compact support} \}$$

where $\mathrm{supp}(f) = \overline{\{x \in X \mid f(x) \neq 0\}}$. Then $C_c(X,\mathbb{C})$ is a complex vector space. For $f \in C_c(X,\mathbb{C})$ we put $\bar{f}(x) = \overline{f(x)}$ (complex conjugate).

Let G be a locally compact grp. We call a linear map $I : C_c(G,\mathbb{C}) \to \mathbb{C}$ a <u>complex Haar integral</u> if the following hold.

(i) $I(f \circ \lambda_a) = I(f)$ for all $a \in G$

(ii) $I(f \cdot \bar{f}) \geq 0$

(iii) $I(f \cdot \bar{f}) = 0$ if and only if $f = 0$

2. <u>Lemma</u> Every locally compact group G admits a complex Haar integral I. If J is another complex Haar integral, then $J = s . I$ for some $s > 0$.

<u>proof</u> <u>Existence</u>. Let I be a Haar integral (cp. §2,2) put $I(f_1 + i f_2) = I(f_1) + i I(f_2)$ when $f_1 = Re(f)$, $f_2 = Im(f)$ are the real and imaginary part of f, respectively. Thus I is $\mathbb{C}$-linear and (i), (ii), (iii) hold (cp. §2.23 for (iii)).

$i = \sqrt{-1}$

<u>Uniqueness</u> If I, J are complex Haar integrals, then their restrictions to $C_c(G,\mathbb{R}) \subseteq C_c(G,\mathbb{C})$ are Haar integrals: if $f \geq 0$, put $h = \sqrt{f}$ so $I(h \cdot h) = I(f) \geq 0$ and if $f \geq 0$ and $f \neq 0$, then $h \neq 0$, whence $I(f) > 0$.

Hence there exists $s > 0$ such that $J(f) = s\,I(f)$ for all $f \in C_c(G,\mathbb{R})$. But then $J(f_1 + if_2)$

$= J(f_1) + iJ(f_2) = s\,I(f_1) + s \cdot i \cdot I(f_2) = s\,I(f_1 + if_2)$

$f_1 = \mathrm{Re}(f)$ $\qquad f \in C_c(G,\mathbb{C})$.

$f_2 = \mathrm{Im}(f)$

$\square$

3. <u>Proposition</u> Let G be a locally compact Lie group. For $f, h \in C_c(G,\mathbb{C})$ we define

$$\langle f | h \rangle = \int_G \overline{f(g)}\, h(g)\, dg$$

Then $\langle - | - \rangle$ is a positive definite hermitian form $C_c(G,\mathbb{C}) \times C_c(G,\mathbb{C}) \longrightarrow \mathbb{C}$.

That is, $\langle - | - \rangle$ is additive in both variables,

$$\langle zf | h \rangle = \bar{z} \langle f | h \rangle \quad , \quad \langle f | zh \rangle = \langle f | h \rangle z ,$$

$$\overline{\langle f | h \rangle} = \langle h | f \rangle \quad , \text{ and } \quad \langle f | f \rangle > 0 \text{ if } f \neq 0 .$$

Proof This follows from the definition. □

4. Def A pre-Hilbert space is a complex vector space E with a positive definite hermitian form $\langle - | - \rangle$ on E.

We put $\|w\| = \sqrt{\langle w | w \rangle}$ and we note that

$$\|u+v\|^2 = \|u\|^2 + 2\,\mathrm{Re} \langle u | v \rangle + \|v\|^2$$

$$\|u+v\|^2 + \|u-v\|^2 = 2 (\|u\|^2 + \|v\|^2) \quad \text{(parallelogram identity)}$$

and $\langle u | v \rangle = \frac{1}{4} \sum_{k=0}^{3} i^k \|u + i^k v\|^2$.

Moreover, $|\langle u | v \rangle| \leq \|u\| \cdot \|v\|$ (Cauchy-Schwarz) with equality if and only if u, v are linearly dependent:

$$\| \langle u | v \rangle u - \|u\|^2 v \|^2 = |\langle v | u \rangle|^2 \cdot \|u\|^2 + \|u\|^4 \|v\|^2 - 2\,\mathrm{Re} (\langle v | u \rangle \langle u | v \rangle) \|u\|^2$$

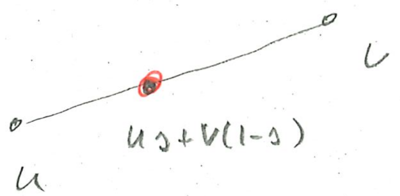
$$= \|u\|^2 (\|u\|^2 \cdot \|v\|^2 - |\langle u | v \rangle|^2) \qquad □$$

In particular, $\|u+v\|^2 \leq (\|u\| + \|v\|)^2$, hence $\| . \|$ is a norm.

If E is complete with respect to $\|\cdot\|$, then E is called a <u>Hilbert space</u>.

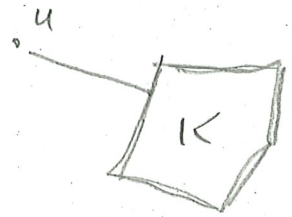
<u>Examples</u>
- $(C_c(G,\mathbb{C}), \langle - | - \rangle)$ is a pre-Hilbert space
- $\mathbb{C}^n$ is a Hilbert space for $\langle u | v \rangle = \sum_{k=1}^{n} \overline{u_k} v_k$

5. <u>Def</u> Let E be a real vector space. A nonempty subset $K \subseteq E$ is <u>convex</u> if for all $u, v \in K$, $s \in [0,1]$ we have $u \cdot s + v(1-s) \in K$



Note: intersections of convex sets are again convex. If $K, L \subseteq E$ are convex, then $K+L$ is convex.
If E is a topological vector space (i.e. $(E,+)$ is a Hausdorff top. group and $E \times \mathbb{R} \to E$, $(w,s) \longmapsto ws$ is continuous), then the closure of a convex set $K \subseteq E$ is again convex: consider for $s \in [0,1]$
$f_s(u,v) = us + v(1-s) \rightsquigarrow f_s(\overline{K \times K}) = f_s(\overline{K} \times \overline{K}) \subseteq \overline{K}$
$\square$

Lemma Let E be a pre-Hilbert space, let $\emptyset \neq K \subseteq E$ be convex and complete. Given $u \in E$, there is a unique $v \in K$ that minimizes the distance to u,



$P_K(u) = v$

Proof put $d = \inf \{ \|u-v\| \mid v \in K \}$.

For $v, w \in E$ we have in general

$$\|(u-v)-(u-w)\|^2 + \|(u-v)+(u-w)\|^2 = 2(\|u-v\|^2 + \|u-w\|^2)$$

$$\leadsto \|v-w\|^2 + 4\|u - \tfrac{1}{2}(v+w)\|^2 = 2\|u-v\|^2 + 2\|u-w\|^2.$$

If $(v_k)_{k\in\mathbb{N}}$ is a sequence in K with $\lim_k \|u-v_k\|^2 = d^2$, then

$$\|v_h - v_\ell\|^2 \leq 2\|u-v_h\|^2 + 2\|u-v_\ell\|^2 - 4d^2$$

$\leadsto$ $(v_k)_{k\in\mathbb{N}}$ is a Cauchy sequence, with $v = \lim_k v_k \in K$.

$\leadsto$ $\|u-v\|^2 = d^2$. If $\|u-v\|^2 = \|u-w\|^2 = d$, then $\|v-w\|^2 = 0$. $\square$

6. Def Let E be a pre-Hilbert space, let $w \in E$, $w \neq 0$, let $s \in \mathbb{R}$. Put

$$H = H(w,s) = \{ u \in E \mid \mathrm{Re}\langle w | u \rangle \leq s \}$$

We call H a half-space and we note that H is closed and convex.

Prop (Banach-Hahn-Mazur)

[S. Banach, Polish math, 1892-1945
H. Hahn, Austr. math. 1879-1934
S. Mazur, Polish math. 1905-1981]

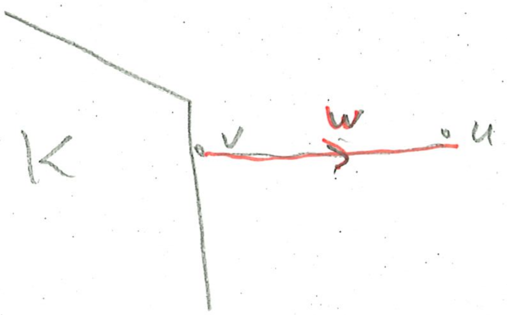
Let E be a Hilbert space, let $K \subseteq E$ be closed and convex. Then

$$K = \bigcap \{ H \subseteq E \mid H \text{ half-space with } K \subseteq H \}$$

Proof The RHS is closed and contains K.

Claim If $u \in E - K$, then there exist a halfspace H with $K \subseteq H$ and $u \notin H$.

Put $v = P_K(u)$ and $w = u - v \leadsto w \neq 0$



Let $s = \mathrm{Re}\langle v | w\rangle$. Then $0 < \mathrm{Re}\langle w | w\rangle$

$= \mathrm{Re}\langle w | u\rangle - s$, whence $u \notin H(w,s)$.

We claim that $K \subseteq H(w,s)$. Let $y \in E - H(w,s)$,

consider $f : t \longmapsto \| (1-t)v + ty - u\|^2$

$$= \|w\|^2 + 2t\,\mathrm{Re}\langle v-y | w\rangle + t^2 \|y - v\|^2$$

with $\frac{d}{dt} f \Big|_{t=0} = 2\,\mathrm{Re}\langle v-y | w\rangle = 2s - \mathrm{Re}\langle y | w\rangle < 0$

Hence there is $t \in [0,1]$ such that

$\|(1-t)v + ty + u\|^2 < \|v - u\|^2$ so $y \notin K$. $\square$

7. Let E be a topological vectorspace.

If $X \subseteq E$ is a subset, then we put

$$\overline{\mathrm{conv}}(X) = \bigcap \left\{ K \subseteq E \;\middle|\; \begin{array}{l} K \text{ is closed and convex,} \\ \text{with } X \subseteq K \end{array} \right\},$$

the __closed convex hull of X__.

__Prop__ Let E be a Hilbert space, let

$C \subseteq E$ be a compact subset. Then

$\overline{\mathrm{conv}}(C) \subseteq E$ is compact.

proof Let $\varepsilon > 0$ and $D = \overline{B}_{\frac{\varepsilon}{2}}(0) \subseteq E$.

Since G is compact, there is a finite set $A \subseteq G$ such that $G \subseteq A + D$. The convex hull of the finite set A is the image of a k-simplex, $k - 1 = |A|$ (wlog $G \neq \emptyset$) hence $\overline{\mathrm{conv}}(A) = L$ is compact. Hence there is $B \subseteq L$ finite with $L \subseteq B + D$. Since L is compact and D closed, $L + D$ is compact, cp §1.11. Moreover, L, D are convex, hence $L + D$ is also convex. Hence

$\overline{\mathrm{conv}}(G) \subseteq L + D$, since

$G \subseteq A + D \subseteq L + D$.

Moreover $L + D \subseteq \underset{\substack{\uparrow \\ \text{finite}}}{B} + D + D \subseteq B + \overline{B}_\varepsilon(0)$

We have shown: for every $\varepsilon > 0$, $\overline{\mathrm{conv}}(G)$ is contained in finitely many ε-balls. This implies that every sequence in $\overline{\mathrm{conv}}(G)$ has a subsequence which is a Cauchy sequence. Since $\overline{\mathrm{conv}}(G)$ is complete, it is compact. ($\to$ Exercise!) $\square$

(The proof holds actually if E is a Banach space).

8. Let E be a pre-Hilbert space, let $X \subseteq E$ be a nonempty subset. Then

$$X^{\perp} = \{ v \in E \mid \langle x | v \rangle = 0 \text{ for all } x \in X \}$$

is a closed (linear) complex subspace of E.

Indeed, $x^{\perp} = \ker(v \longmapsto \langle x | v \rangle)$ is closed and

$$X^{\perp} = \bigcap_{x \in X} x^{\perp}.$$

<u>Lemma</u> Suppose that $F \subseteq E$ is a complete complex (linear) subspace of the pre-Hilbert space E. Then $E = F \oplus F^{\perp}$

<u>proof</u> We define $P_F : E \to F$ as in Lemma §3.5 (the nearest point projection) and $u_1 = p(u)$, $u_2 = u - u_1$. We claim that $u_2 \in F^{\perp}$.

For every $w \in F$, $w \neq 0$ and every $z \in \mathbb{C}$ we have

$$\|u_2\|^2 \leq \|u_2 + zw\|^2 = \|u_2\|^2 + 2\,\mathrm{Re}(\langle u_2 | w \rangle z) + \|w\|^2 |z|^2$$

For $z = \frac{-1}{\|w\|^2} \langle w | u_2 \rangle$ we have therefore

$$0 \leq \frac{-1}{\|w\|^2} |\langle w | u_2 \rangle|^2$$

$\rightsquigarrow \langle w | u_2 \rangle = 0$, $u_2 \in F^{\perp}$.

Since $u_1 \in F$, we have $u = u_1 + u_2 \in F + F^\perp$.
But $F \cap F^\perp = \{0\}$ (since $\langle - | - \rangle$ is positive definite), whence $E = F \oplus F^\perp$ $\square$ #

9. Lemma A pre-Hilbert space E is locally compact if and only if E has finite dimension.

proof If $\dim(E) = m < \infty$, then $E \cong \mathbb{C}^m$ is locally compact.

Suppose that $\dim(E) = \infty$. Inductively, we find vectors $u_1, \dots, u_m \in E$ with $\|u_k\| = 1$ and $\langle u_j | u_k \rangle = 0$ for $j \neq k$. Indeed, if $u_1, \dots, u_m$ are constructed, put $F = \operatorname{span}_{\mathbb{C}}\{u_1, \dots u_m\}$ as $F \cong \mathbb{C}^m$ is complete, then $E = F \oplus F^\perp$ by §3.8. Now choose $u_{m+1} \in F^\perp$ etc.

For $k \neq j$ we have $\|u_j - u_k\|^2 = 2$, hence $(u_k)_{k=0}^{\infty}$ has no convergent subsequence.

Hence $\overline{B}_1(0) \subseteq E$ is not compact. But then $\overline{B}_\varepsilon(0) = \varepsilon \cdot \overline{B}_1(0)$ is also not compact for any $\varepsilon > 0$. Hence E is not locally compact.

10. Def Let E, F be normed complex vector spaces. A linear map $T: E \to F$ is called an operator. An operator T is called bounded if for every bounded set $X \subseteq E$, the set $T(X) \subseteq F$ is bounded.

[Warning: this does not mean that $T(E) \subseteq F$ is bounded!!!]

Prop Let E, F be normed vector spaces, let $T: E \to F$ be an operator. The following are equivalent:

(i) T is continuous.
(ii) T is bounded.
(iii) $T(B_1(0)) \subseteq F$ is bounded.

proof (i) => (ii): choose $\varepsilon > 0$ so that $T(B_\varepsilon(0)) \subseteq B_1(0)$. Then $T(B_r(0)) \subseteq B_{r/\varepsilon}(0)$ for all $r > 0$, hence T is bounded.

(ii) $\Rightarrow$ (iii): obvious.

(iii) $\Rightarrow$ (i): choose $r>0$ so that $T(B_1(0)) \subseteq B_r(0)$. For $\varepsilon>0$ we have then $T(B_{\varepsilon/r}(0)) \subseteq B_\varepsilon(0)$, hence T is continuous at 0, hence continuous by Exercise 1.1. $\square$

11. <u>Def</u> If $T: E \to F$ is a bounded operator, we put $\|T\|_F = \sup\{\|Tu\|_F \mid u \in B_1(0) \subseteq E\}$ ($\|.\|_F$ is the norm on F, $\|.\|_E$ is the norm on E). Thus $\|Tv\|_F \leq \|T\| \cdot \|v\|_E$ and in particular $\|Tv - Tu\|_F \leq \|T\| \cdot \|v-u\|_E$. A bounded operator T is thus $\|T\|$- <u>Lipschitz continuous</u>. The number $\|T\|$ is called the <u>operator norm</u> of T. If we put

$$B(E,F) = \{T: E \to F \mid T \text{ is a bounded operator}\}$$

then $B(E,F)$ is a complex vector space and $\|T\|$ is a norm on this vector space.

For $u, v \in E$, $T, S \in \mathcal{B}(E,F)$ we have

$$\|(S+T)(u+v)\|_F \le \|S\|\cdot\|u\|_E + \|S\|\cdot\|v\|_E + \|T\|\cdot\|u\|_E + \|T\|\cdot\|v\|_E .$$

If $E \xrightarrow{T} F \xrightarrow{S} H$ are bounded operators, then $S\circ T : E \to E$ is bounded with $\|S\circ T\| \le \|S\|\cdot\|T\|$.

All these claims are easy <u>exercises</u>.

<u>Recall</u>: A <u>Banach space</u> is a complete normed vector space. Every Hilbert space is a Banach space.

<u>Prop</u> Let E, F be normed vector spaces. If F is complete, then $\mathcal{B}(E,F)$ is also complete. In particular, the dual space $E^* = \mathcal{B}(E,\mathbb{C})$ of any normed vector space is complete.

<u>Proof</u> Let $(T_k)_{k\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{B}(E,F)$ and let $u \in E$. Since

$\|T_k u - T_\ell u\| \le \|T_k - T_\ell\|\cdot\|u\|$, $(T_k u)_{k\in\mathbb{N}}$ is a Cauchy sequence in F. Put $Tu = \lim_k T_k u$.

For $u,v \in E$ and $z \in \mathbb{C}$ we have

$T_k u + z T_k v - T_k(u+zv) = 0$, hence

$Tu + zTv - T(u+zv) = 0 \Rightarrow T$ is linear.

Choose m so that $\|T_k - T_l\| \le 1$ for $k,l \ge m$.

For $u \in B_1(0) \subseteq E$ we have $\|T_k u - T_m u\| \le 1$

$k \ge m$

hence $\|Tu - T_m(u)\| \le 1 \Rightarrow T - T_m \in \mathcal{B}(E,F) \Rightarrow$

$T \in \mathcal{B}(E,F)$, T is bounded.

Finally, let $m \in \mathbb{N}$ so that $\|T_k - T_l\| \le \varepsilon$ for $k,l \ge m$.

Then (1) $\|T - T_l\| \le \varepsilon$ for all $l \ge m$, hence $\lim_l \|T - T_l\| = 0$ $\square$

<u>Lemma</u> Let E,F be normed vector spaces. If $\dim(E) < \infty$, then every linear map $T: E \to F$ is bounded.

<u>proof</u> Let $u_1, \dots, u_m \in E$ be a basis. Put

$S: \mathbb{C}^m \to E$, $S(z_1, \dots, z_m) = z_1 u_1 + \dots + z_m u_m$.

Then S and $T \circ S: (z_1, \dots, z_m) \mapsto z_1 T u_1 + \dots + z_m T u_m$

are continuous, hence bounded. Let

$\mathbb{S}^{2m-1} = \{ w \in \mathbb{C}^m \mid |w_1|^2 + \dots + |w_m|^2 = 1 \} \Rightarrow$

$S(\mathbb{S}^{2m-1}) \subseteq E$ is compact. Put $s = \min\{ \|Sw\| \mid$

$w \in \mathbb{S}^{2m-1} \}$. Then $s > 0$, hence

$\underbrace{\bar{B}_s(0)}_{\subseteq E} \subseteq S(\underbrace{\bar{B}_1(0)}_{\subseteq \mathbb{C}^m}) \Rightarrow S^{-1}$ is continuous

(1) For $\|u\| \le 1$, $\|Tu - T_l u\| \le \varepsilon \|u\| \le \varepsilon$

hence $\|T - T_l\| \le \varepsilon$

$\rightsquigarrow T = T \circ S \circ S^{-1}$ is continuous. $\square$

12. <u>Prop</u> Let $(E, \|\cdot\|)$ be a normed vector space. Then there exist a Banach space (a complete normed vector space) $(\hat{E}, \|\cdot\|)$, a linear map $j : E \to \hat{E}$ which is isometric ($\|ju\| = \|u\|$) with the following universal property: given a bounded operator $T : E \to F$, where F is a Banach space, there is a unique extension $\hat{T}$

$$\begin{tikzcd} E \arrow[r, "T"] \arrow[d, "j"'] & F \\ \hat{E} \arrow[ur, "\hat{T}"'] & \end{tikzcd}$$

with $\hat{T}$ bounded. Moreover, $j(E) \subseteq \hat{E}$ is dense.

Then $j : E \to \hat{E}$ is called the <u>completion</u> of E.

<u>Proof</u> Let $G \subseteq E^{\mathbb{N}}$ denote the set of all Cauchy sequences in E, let $N \subseteq G$ denote the sequences converging to 0. Then G, N are complex vector spaces. Put $\hat{E} = G/N$

For $(u_k)_{k\in\mathbb{N}} \in G$, $(\|u_k\|)_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. Put $\|(u_k)_{k\in\mathbb{N}}\| = \lim_k \|u_k\|$. For $(v_k)_{k\in\mathbb{N}} \in N$ we have $\|(u_k+v_k)_{k\in\mathbb{N}}\| = \|(u_k)_{k\in\mathbb{N}}\|$ hence we have a well-defined norm on $\hat{E} = G/N$ given by $\|(u_k)_{k\in\mathbb{N}} + N\| = \|(u_k)_{k\in\mathbb{N}}\|$. We define $j(u) = (u)_{k\in\mathbb{N}}$ $\Rightarrow$ $j: E \to \hat{E}$ is isometric and linear. #

Let $(u_k)_{k\in\mathbb{N}} \in G$. Given $\varepsilon > 0$, there is $m \geq 0$ such that $\|u_k - u_l\| \leq \varepsilon$ for $j, l \geq m$. Hence $\|(u_k)_{k\in\mathbb{N}} - (u_m)_{k\in\mathbb{N}}\| \leq \varepsilon$ $\Rightarrow$ $j(E) \subseteq \hat{E}$ is dense and every Cauchy sequence in $j(E)$ has a limit in $\hat{E}$. Hence $\hat{E}$ is complete.

Finally, let $T: E \to F$ be bounded. Given $(u_k)_{k\in\mathbb{N}} \in G$, put $\hat{T}((u_k)_{k\in\mathbb{N}} + N) = \lim_k Tu_k \in F$. Then $\hat{T}$ is linear and bounded. ... $\square$

<u>Cor</u> Let $(E, \langle\,|\,\rangle)$ be a pre-Hilbert space. Then $\hat{E}$ is a Hilbert space.

<u>proof</u> We define $\langle\,|\,\rangle$ on $\hat{E}$ via

$$\langle u|v\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k \|u + i^k v\|^2$$

Then $\langle\,|\,\rangle$ is continuous on $\hat{E}\times\hat{E}$. Since $j(E)\subseteq\hat{E}$ is dense, $\langle\,|\,\rangle$ is a positive definite hermitian form.

For example $\langle j(u)|j(u)\rangle \geq 0 \Rightarrow \langle w|w\rangle \geq 0$ on $\hat{E}\times\hat{E}$ etc. $\square$

13. <u>Observation</u> If E is a topological vector space and if $H\subseteq E$ is a linear subspace, then $\overline{H}\subseteq E$ is also a linear subspace. Indeed, $\overline{H}\subseteq E$ is a subgroup by §1.5. For $z\in\mathbb{C}$ we have $z\cdot\overline{H}\subseteq\overline{H}$, whence $\overline{H}$ is a linear subspace.

<u>Theorem</u> (Riesz Representation Theorem)

[F. Riesz, Hungarian math., 1880–1956]

Let $(E, \langle\cdot|\cdot\rangle)$ be a Hilbert space. For $u\in E$ put

$$L(u) = \langle u|-\rangle : E\to\mathbb{C}$$
$$L(u)(v) = \langle u|v\rangle$$

Then $L: E\to E^* = B(E,\mathbb{C})$ is a semi-linear isomorphism which is an isometry.

<u>Proof</u> $L(zu)(v) = \langle zu|v\rangle = \bar{z}\langle u|v\rangle$

$\Rightarrow$ L is semi-linear. For $u\neq 0$ we have

$L(u)(u) = \|u\|^2 \neq 0 \Rightarrow$ L is <u>injective</u>.

Let $\xi: E \to \mathbb{C}$ be a continuous linear map, $\xi \neq 0$.

Put $H = \ker(\xi)$. Then $E/_H \cong \mathbb{C}$ by the Homomorphism Theorem, hence H has codimension 1, and $E = H \oplus H^\perp$ by §3.8, with $H^\perp \cong \mathbb{C}$. Pick $w \in H^\perp$, $w \neq 0$ and put $u = \overline{\xi(w)} \frac{1}{\|w\|^2} w$. Then

$$\xi(u) = \overline{\xi(w)} \frac{1}{\|w\|^2} \xi(w) = |\xi(w)|^2 \frac{1}{\|w\|^2}$$

$$\langle u|u\rangle = |\xi(w)|^2 \frac{\|w\|^2}{\|w\|^4} = \xi(u)$$

$\Rightarrow \xi(v) = \langle u|v\rangle$ for all $v \in E$

Thus $L(u) = \xi$, and L is <u>surjective</u>.

If $\|v\| \leq 1$, then $|L(u)(v)| \leq \|u\| \cdot \|v\| \leq \|u\|$, whence $\|L(u)\| \leq 1$. If $u \neq 0$, then $\left|L(u)\left(\frac{1}{\|u\|}u\right)\right| = \frac{|\langle u|u\rangle|}{\|u\|}$ $= \|u\| \Rightarrow \|L(u)\| \geq \|u\| \Rightarrow \|L(u)\| = \|u\|$ $\square$

14. <u>Prop</u> Let $(E, \langle -|-\rangle)$ be a Hilbert space, let $b: E \times E \to \mathbb{C}$ be a sesquilinear form

i.e. b is additive in both variables, and

$b(zu,v) = \bar{z}\, b(u,v)$ $\qquad b(u,zv) = z\, b(u,v)$.

The following are equivalent:

(i) b is continuous.

(ii) there is $r \geq 0$ such that $|b(u,v)| \leq r \|u\| \cdot \|v\|$ for all $u,v \in E$

(iii) There is $T \in B(E,E)$ such that

$$b(u,v) = \langle Tu \mid v \rangle \quad \text{for all } u,v \in E.$$

If these conditions hold, then $\|T\| \leq r$, and T is determined uniquely by b.

proof (i) => (ii): Let $\varepsilon > 0$ such that

$|b(u,v)| \leq 1$ for all $\|u\|, \|v\| \leq \varepsilon$.

Then $|b(u,v)| \leq \|u\| \cdot \|v\| \frac{1}{\varepsilon}$ holds for all $u,v \in E$.

(ii) => (iii): consider the map $b(v,-)$, for $v \in E$.

By §3.13, there is a unique $w \in E$ such that $\langle w \mid - \rangle = b(v,-)$. Put $Tv = w$. Hence

$$L(w) = LTv = \langle Tv \mid - \rangle = b(v,-)$$

Hence T is linear. Moreover, we have for $\|v\| \leq 1$

that
$$\begin{aligned} \langle Tv \mid Tv \rangle &= \|Tv\|^2 \\ \| \\ b(Tv,v) &\leq \|Tv\| \cdot \|v\| \cdot r \leq \|Tv\| \cdot r, \text{ whence} \end{aligned}$$

$\|Tv\| \leq r \leadsto \|T\| \leq r$.

(iii) => (i) is clear.

If $T' \in B(E,E)$ with $b(u,v) = \langle T'u \mid v \rangle$,
$= L(T'u)(v)$
$= L(Tu)(v)$

then $LT' = LT$, but L is bijective $\leadsto T = T'$. $\square$

<u>Cor</u> If E is a Hilbert space and $T : E \to E$ is a bounded operator, then there is a unique bounded operator $T^* : E \to E$ such that
$\langle u | Tv \rangle = \langle T^* u | v \rangle$ holds for all u, v.
Moreover $\|T\| = \|T^*\|$, and $T^{**} = T$.
We call T^* the <u>adjoint operator</u> of T.

<u>proof</u> $b(u,v) = \langle u | Tv \rangle$ is continuous and sesquilinear,
hence $\langle u | Tv \rangle = \langle T^* u | v \rangle$ for a unique T^*.
Also, $T^{**} = T$, and $|\langle T^* u | v \rangle| = |\langle u | Tv \rangle|$
$\leq \|u\| \cdot \|Tv\| \leq \|u\| \cdot \|v\| \, \|T\| \leadsto \|T^*\| \leq \|T\|$
Conversely, $\|T^{**}\| = \|T\| \leq \|T^*\|$. $\square$

We call T <u>self-adjoint</u> if $T = T^*$.

<u>Example</u> $E = \mathbb{C}^m$, $\langle u | v \rangle = \sum_{k=1}^{m} \overline{u_k} v_k$

Then $T : \mathbb{C}^m \to \mathbb{C}^m$ is given by a matrix $(a_{k,l})_{k,l=1}^{m}$, and T^* is given by $(\overline{a_{l,k}})_{k,l=1}^{m}$
(conjugate-transpose matrix).

Note also: if $T = T^*$, then
$\langle u | Tu \rangle = \langle Tu | u \rangle = \overline{\langle u | Tu \rangle}$, hence
$\langle u | Tu \rangle \in \mathbb{R}$.

15. <u>Lemma</u> Let E be a Hilbert space, let $T \in B(E,E)$ be self-adjoint. Then

$$\|T\| = \sup\{\, |\langle Tu|u\rangle| \mid \|u\| \leq 1 \,\}$$ #

<u>proof</u> Put $\nu(T) = \sup\{\, \underbrace{|\langle Tu|u\rangle|}_{\leq \|T\|\cdot\|u\|\cdot\|u\| \leq \|T\|} \mid \|u\| \leq 1 \,\}$

$\Rightarrow \nu(T) \leq \|T\|$, and $|\langle Tu|u\rangle| \leq \nu(T)\|u\|^2$.

Now $\langle T(v+w)|v+w\rangle - \langle T(v-w)|v-w\rangle = 4\,\mathrm{Re}\langle Tv|w\rangle$

Put $s>0$, $v = su$, $w = \frac{1}{s}Tu \Rightarrow \langle Tv|w\rangle = \|Tu\|^2$,

whence $4\|Tu\|^2 \leq \nu(T)\left(\|su + \frac{1}{s}Tu\|^2 + \|su - \frac{1}{s}Tu\|^2\right)$

$$= 2\cdot\nu(T)\left(s^2\|u\|^2 + \frac{1}{s^2}\|Tu\|^2\right)$$

If $Tu \neq 0$ put $s = \sqrt{\dfrac{\|Tu\|}{\|u\|}}$

$\Rightarrow 4\|Tu\|^2 \leq 2\,\nu(T)\cdot 2\,\|Tu\|\cdot\|u\|$

$\Rightarrow \|Tu\| \leq \nu(T)\|u\|$. This is also true if $Tu = 0$.

Hence $\|T\| \leq \nu(T)$. □

16. <u>Def</u> Let E, F be normed vector spaces. An operator $T: E \to F$ is called <u>compact</u> if for every bounded set $X \subseteq E$, the image $T(X)$ has compact closure in F.

Since compact sets are bounded, every compact operator is bounded (i.e. continuous).

If $\dim E < \infty$, then every operator $T: E \to F$ is compact. ⊛ But in general, bounded operators are not compact.

⊛ Or if $\dim(F) < \infty$, then every bounded $T: E \to F$ is compact.

<u>Example</u> E Hilbert space, $F \subseteq E$ finite dimensional.

$\leadsto E = F \oplus F^{\perp}$ by §3.8. The projection

$P_F : E \to F$, $u + v \mapsto u$ $\quad \begin{matrix} u \in F \\ v \in F^{\perp} \end{matrix}$

is compact.

17. Let E be a Hilbert space, let $T \in \mathbb{B}(E, E)$ be self-adjoint. Let $\sigma_p(T)$ denote the set of all eigenvalues of T, $\sigma_p(T) \subseteq \mathbb{C}$.

For $z \in \sigma_p(T)$ put $E_z = \{ u \in E \mid Tu = zu \}$
$= \ker(z\mathbb{1} - T)$

18. <u>Theorem</u> (The spectral theorem for compact self-adjoint operators).

Let $E \neq 0$ be a Hilbert space, let $T \in B(E, E)$ be compact and self-adjoint. Then the following hold.

(i) $\sigma_p(T) \subseteq \mathbb{R}$, all eigenvalues are real.
For $s, t \in \sigma_p(T)$ with $s \neq t$ we have
$$E_s \perp E_t$$

(ii) $\|T\| \in \sigma_p(T)$ or $-\|T\| \in \sigma_p(T)$

(iii) $$\overline{\sum_{s \in \sigma_p(T)} E_s} = E$$

(iv) $\sigma_p(T) \subseteq \mathbb{R}$ is countable (or finite).
If $s \in \mathbb{R}$ is an accumulation point of $\sigma_p(T)$, then $s = 0$.

(v) If $s \in \sigma_p(T)$, $s \neq 0$, then $\dim E_s < \infty$.

<u>Proof</u> If $T = 0$, we have nothing to show so assume that $T \neq 0$.

(i): If $\underset{u \neq 0}{Tu = zu}$, then $\langle Tu|u\rangle = z\langle u|u\rangle \in \mathbb{R} \Rightarrow z \in \mathbb{R}$

If $s, t \in \sigma_p(T)$, $s \neq t$, $u \in E_t$, $v \in E_s$

$$\Rightarrow \quad t\langle u|v\rangle = \langle Tu|v\rangle = \langle u|Tv\rangle = s\langle u|v\rangle$$
$$\Rightarrow \langle u|v\rangle = 0$$

(ii): By § 3.15 there is a sequence $(u_n)_{n\in\mathbb{N}}$

$\|u_n\| \le 1$, $\lim_n |\langle Tu_n | u_n\rangle| = \|T\| \ne 0$.

Passing to a subsequence, we may assume that $\lim_n u_n = v$ exists (since T is compact), and that $t = \lim_n \langle Tu_n | u_n\rangle = c \cdot \|T\| \ne 0$, $c = \pm 1$.

$\leadsto t \ne 0$, $|t| = \|T\|$.

$$0 \le \|Tu_n - tu_n\|^2 = \|Tu_n\|^2 - 2\langle Tu_n | tu_n\rangle + t^2 \|u_n\|^2$$
$$\le \underbrace{2t^2 - 2t\langle Tu_n | u_n\rangle}_{\text{converges to } 0}$$

$\leadsto \lim_n tu_n = \lim_n Tu_n = v \leadsto \lim_n u_n = \frac{1}{t} v$,

$Tu = v = tu \leadsto t \in \sigma_p(T)$ and $|t| = \|T\|$.

(iii) If $H \subseteq E$ is a T-invariant linear subspace, then $H^\perp$ is also T-invariant: $u \in H$, $v \in H^\perp$

$\leadsto \langle Tu | v\rangle = \langle u | Tv\rangle = 0$

Now $H = \sum_{t\in\sigma_p(T)} E_t \subseteq E$ is T-invariant,

hence $\overline{H} \subseteq E$ is T-invariant, hence $\overline{H}^\perp$ is T-invariant. Then $T|_{\overline{H}^\perp} : \overline{H}^\perp \to \overline{H}^\perp$

is a compact selfadjoint operator. If $\overline{H}^{\perp} \neq 0$, then there are eigenvalues ↯. Hence $\overline{H}^{\perp} = 0$. Thus $E = \overline{H}$ by § 3, 8.

(iv) and (v): For $\varepsilon > 0$ put $S_\varepsilon = \{ t \in \sigma_p(T) \mid |t| \geq \varepsilon \}$ and $H = \overline{\sum_{t \in S_\varepsilon}} E_t$.

<u>Claim : $\dim(H) < \infty$.</u>

Otherwise, there would be a sequence $(u_n)_{n \in \mathbb{N}}$ with $\|u_n\| = 1$, $u_n \in E_{s_n}$ for $s_n \in S_\varepsilon$, $u_{n+1} \perp \{u_0, \dots, u_n\}$. Then for $k \neq l$

$$\|Tu_k - Tu_l\|^2 \geq 2 \cdot \varepsilon^2 \quad (\|Tu_k - Tu_l\|^2 = s_k^2 + s_l^2)$$

↯. Hence $\dim(H) < \infty$. In particular, S_ε is finite and $\dim E_s < \infty$ for all $s \in S_\varepsilon$. $\square$

19. <u>Def</u> Let E be a Hilbert space. A linear operator $T : E \to E$ is called <u>unitary</u> if T is surjective and if $\|Tu\| = \|u\|$ for all $u \in E$. Thus T is bounded. The unitary operators form a group $U(E)$, the <u>unitary group</u>.

Let G be a topological group. A continuous action $G \times E \to E$ is called a <u>unitary representation</u> if every $g \in E$ acts as a unitary operator on E. We call E a <u>Hilbert-G-module</u>. #

<u>Example</u> $E = \mathbb{C}^m$, $\langle u | v \rangle = \sum_{k=1}^{m} \bar{u}_k v_k$

$$U(E) = \{ g \in \mathbb{C}^{m \times m} \mid g^* g = \mathbb{1} \} = U(m)$$

when $(g_{k,l})^*_{k,l} = (\overline{g_{l,k}})_{k,l}$

20. If G is a locally compact group, we put

$L^2(G, \mathbb{C}) = \widehat{C_c(G, \mathbb{C})}$, the completion w.r.t.

$$\langle f, h \rangle = \int_G \bar{f}(g) h(g) \, dg \,, \quad \| f \|_2 = \left(\int_G \bar{f}(g) f(g) \, dg \right)^{\frac{1}{2}}$$

For $g \in G$ we put $T_g f = f \circ \lambda_{g^{-1}}$.

Then $T_g \in U(E)$. Note that for $a, b \in G$ $T_a \circ T_b = T_{ab}$. Note: the use of $\lambda_{g^{-1}}$ makes sure we have a <u>left</u> action.

Lemma The action of G on $C_c(G,\mathbb{C})$ is faithful, i.e. $T_g \equiv id \Rightarrow g = e$.

proof Let $g \in G$, $g \neq e$. Let $U \subseteq G$ be an identity neighbourhood with $UU^{-1} \subseteq G - \{g\} \Rightarrow g \notin UU^{-1}$ $\Rightarrow gU \cap U = \emptyset$. Let $\varphi \in C_c(G)$, $\varphi \geq 0$, $\varphi \neq 0$ with $supp(\varphi) \subseteq U$. Then $\varphi \circ \lambda_{g^{-1}} \neq \varphi$, because $supp(\varphi \lambda_{g^{-1}}) = g\, supp(\varphi)$. □

Theorem Let G be a locally compact group. Then the map

$$G \times L^2(G,\mathbb{C}) \to L^2(G,\mathbb{C})$$
$$(g,\varphi) \longmapsto T_g\varphi$$

is a (left) Hilbert module for G.

proof For every $\varphi \in C_c(G,\mathbb{C})$ we have $\|T_g\varphi\|_2 = \|\varphi\|_2$. Hence $T_{g^{-1}}$ extends to a bounded operator $\hat{T}_g : L^2(G,\mathbb{C}) \to L^2(G,\mathbb{C})$, cp. §3.12 which we also denote by T_g. Then $\|T_g\varphi\|_2 = \|\varphi\|_2$ for all $\varphi \in L^2(G,\mathbb{C})$.

<u>Claim</u> Given $\varepsilon > 0$, $\psi \in L^2(G, \mathbb{C})$, there is an identity neighbourhood $V \subseteq G$ such that the following holds: for all $v \in V$, $\varphi \in L^2(G, \mathbb{C})$, $g \in G$ with $\|\varphi - \psi\|_2 \le \frac{\varepsilon}{4}$ we have $\| T_g \psi - T_{gv} \varphi \| \le \varepsilon$.

In particular, $G \times L^2(G, \mathbb{C}) \to L^2(G, \mathbb{C})$ is continuous.

<u>proof of the claim.</u> Choose $\rho \in C_c(G, \mathbb{C})$ with $\|\psi - \rho\|_2 \le \frac{\varepsilon}{4}$, put $C = \mathrm{supp}(\rho)$. Let $U \subseteq G$ be an identity neighbourhood with $\overline{U}$ compact, let $\eta : G \to [0,1]$ be continuous with compact support, such that $\eta(x) = 1$ for all $x \in \overline{U} C$, cp. §2.14. Let $V \subseteq G$ be a symmetric identity neighbourhood such that for all $v \in V$, $x \in G$ we have

$$| \rho(x) - \rho(v^{-1}x) | \cdot \|\eta\|_2 \le \frac{\varepsilon}{4}$$

Then $| \rho(x) - \rho(v^{-1}x) | \cdot \|\eta\|_2 \le \frac{\varepsilon}{4} \cdot \eta(x)$, because:

- this is true for $x \in \overline{U} C$
- for $x \notin \overline{U} C \Rightarrow x \notin C$ and $v^{-1}x \notin C \Rightarrow$ LHS $= 0$.

Integration yields

$$\int_G | \rho(g) - \rho(v^{-1}g) |^2 \cdot \|\eta\|_2^2 \, dg \le \left(\frac{\varepsilon}{4}\right)^2 \|\eta\|_2^2$$

$$\Rightarrow \| \rho - T_v \rho \|_2 \le \frac{\varepsilon}{4} .$$

If $\|\varphi-\psi\|_2 \leq \frac{\varepsilon}{4}$, then $\|\varphi - \rho\|_2 \leq \frac{\varepsilon}{2}$, whence

$$\|T_g\psi - T_{gv}\varphi\|_2 = \|\psi - T_v\varphi\|_2 \leq \|\psi-\rho\|_2 + \|\rho - T_v\rho\|_2 + \underbrace{\|T_v\rho - T_v\varphi\|_2}_{=\|\rho-\varphi\|_2}$$

$$\leq \varepsilon$$

$\square$

This shows that $L^2(G,\mathbb{C})$ is a Hilbert-G-module.

$\square$

Note that the action of G on $L^2(G,\mathbb{C})$ is faithful, since $C_c(G,\mathbb{C})$ is dense in $L^2(G,\mathbb{C})$.

Convention. If G is compact, we normalise the Haar integral such that $\int_G 1\,dg = 1$.

("G has volume 1").

21. Theorem Let G be a compact group, $\int_G dg = 1$. Let E be a Hilbert-G-module, let $S \in B(E,E)$. Then there is a unique $\tilde{S} \in B(E,E)$ such that

$$\langle \tilde{S}u | v\rangle = \int_G \langle Sgu | gv\rangle\, dg \qquad \text{for all } u,v \in E.$$

Moreover, $\|\tilde{S}\| \leq \|S\|$ and $\tilde{S}a = a\tilde{S}$ for all $a \in G$.

If S is selfadjoint, then $\tilde{S}$ is selfadjoint.
If S is compact, $\tilde{S}$ is compact.

<u>proof</u> The map $g \mapsto \langle Sgu | gv \rangle$ is continuous, hence in $C_c(G, \mathbb{C})$. Put $b(u,v) = \int_G \langle Sgu | gv \rangle dg$.

Then b is sesquilinear, $b: E \times E \to \mathbb{C}$.

By Cauchy-Schwarz, $|\langle Sgu | gv \rangle| \leq \|S\| \cdot \|u\| \cdot \|v\|$, hence $|b(u,v)| \leq \int_G \|S\| \cdot \|u\| \cdot \|v\| \, dg = \|S\| \cdot \|u\| \cdot \|v\|$.

Thus b is continuous by §3.14, hence $\tilde{S}$ exist uniquely by §3.14 such that $b(u,v) = \langle \tilde{S}u | v \rangle$.
Moreover, $\|\tilde{S}\| \leq \|S\|$. For $a \in G$ we have

$$\int_G \langle Sgau | gav \rangle dg \underset{\text{Invariance}}{=} \int_G \langle Sgu | gv \rangle dg$$

hence $b(u,v) = b(au, av)$

$$\langle \tilde{S}u | v \rangle = \langle \tilde{S}au | av \rangle = \langle a^{-1}\tilde{S}au | v \rangle$$

$\leadsto a^{-1}\tilde{S}a = S$.

If S is selfadjoint, then $b(u,v) = \overline{b(v,u)}$

$\leadsto \tilde{S}^* = \tilde{S}$. $\square$

Suppose S is compact. Put $W = \overline{S(\overline{B}_1(0))} \subseteq E$

$\Rightarrow W$ is compact. Then $GW = \{ gw \mid g \in G, w \in W \}$ is

compact, and so is $K = \overline{\mathrm{conv}}(GW)$ by § 3.7.

Let $H = H(w, s) = \{ u \in E \mid \underbrace{\mathrm{Re}\langle w | u \rangle}_{= \mathrm{Re}\langle u | w \rangle} \leq s \}$ be a halfspace,

with $K \subseteq H$. For $\|u\| \leq 1$, $g \in G$ we have

$g^{-1} S g u \in GW \subseteq K$, hence

$$\mathrm{Re}\langle \tilde{S} u | w \rangle = \mathrm{Re} \int_G \langle S g u | g w \rangle \, dg$$

$$= \int_G \mathrm{Re}\langle S g u | g w \rangle \, dg = \int_G \mathrm{Re}\langle g^{-1} S g u | w \rangle \, dg$$

$$\leq \int_G s \, dg = s,$$

whence $\tilde{S} u \in H(w, s)$. Thus $\tilde{S} u \in K$

by § 3.6 (Banach-Hahn-Mazur), i.e.

$\tilde{S}(\overline{B}_1(0)) \subseteq K$, hence $\tilde{S}$ is compact. $\square$

22. <u>Def</u> Let G be a topological group. A Hilbert-G-module $E \neq \{0\}$ is called <u>irreducible</u> if or <u>simple</u> if $\{0\}, E$ are the only subspaces of E which are Hilbert-G-modules.

<u>Lemma</u> Let G be a compact group, $E \neq \{0\}$ a Hilbert-G-module. Then E contains a finite dimensional simple Hilbert-G-module $F \subseteq E$.

<u>Proof</u> Let $w \in E$, $w \neq 0$, put $T(u) = \langle w|u\rangle w$.

Then T is selfadjoint and compact:

$\langle Tu|v\rangle = \langle\langle w|u\rangle w|v\rangle = \langle u|w\rangle\langle w|v\rangle =$
$\langle u|Tv\rangle = \langle u|\langle w|v\rangle w\rangle = \langle u|w\rangle\langle w|v\rangle$

$\|u\| \leq 1 \rightsquigarrow \|Tu\| \leq \|w\|^2$ and $Tu \in \mathbb{C}w$

$\rightsquigarrow T(\overline{B}_1(0)) \subseteq \mathbb{C}w$ is compact.

Consider $\tilde{T}$ as in §3.21. Then $\tilde{T}$ is compact and selfadjoint. Moreover, $\langle Tgw|gw\rangle = \langle\langle w|gw\rangle w|\underset{\geq 0}{gw}\rangle$

$= \underbrace{\langle w|gw\rangle\overline{\langle w|gw\rangle}}_{\geq 0} > 0$ for $g = e$, hence $\langle\tilde{T}w|w\rangle > 0$

$\rightsquigarrow \tilde{T} \neq 0$. By the spectral theorem §3.18, $\tilde{T}$ has a finite dimensional eigenspace E_λ, $\lambda \neq 0$.

Since $a\tilde{T} = \tilde{T}a$ for all $a \in G$, $aE_\delta \subseteq E_\delta$,

let now $F \subseteq E_\delta$ be a G-invariant subspace of minimal positive dimension. Then F is simple. $\square$

Cor If G is a compact group, then every simple Hilbert-G-module has finite dimension.

24. Theorem Let E be a Hilbert-G-module for a compact group G, $E \neq \{0\}$. Then there is a family $(F_i)_{i \in I}$ of simple G-modules $F_i \subseteq E$, such that $F_i \perp F_j$ for $i \neq j$, with

$$\overline{\sum_{i \in I} F_i} = E$$

proof Let $\mathfrak{F}$ denote the set of all simple G modules $F \subseteq E$. Let K be a set of cardinality $|K| > |\mathfrak{F}|$, let P denote the set of all maps $F : I \to \mathfrak{F}$, with $I \subseteq K$, and $F_i \perp F_j$ for $i \neq j$. We order P partially by inclusion, $F' \geq F$ if $I' \supseteq I$ and $F'|_I = F$

$$\begin{tikzcd} I' \arrow[r, "F'"] & \mathfrak{F} \\ I \arrow[u, phantom, "\cup"] \arrow[ur, "F"'] & \end{tikzcd}$$

Every linearly ordered subset of P has an upper bound (the union), hence P contains maximal elements by Zorn's Lemma (note: $P \neq \emptyset$ by lemma §3.22). Let $F: I \to \mathcal{F}$ be maximal, put $D = \overline{\sum_{i \in I} F_i}$. If $D \neq E$, then $D^{\perp} \neq \{0\}$ and $E = D \oplus D^{\perp}$. Then $D^{\perp}$ contains an $F' \in \mathcal{F}$ by §3.22. Choose $k \in K - I$, put $I' = I \cup \{k\}$, $F_k = F'$ ↯ Hence $D^{\perp} = \{0\}$.

$\square$

25. <u>Observation</u> If E is a finite-dimensional Hilbert-G-module, $\dim(E) = m$, then $E \cong \mathbb{C}^m$ as a Hilbert space (choose an orthonormal basis using Gram-Schmidt) and we obtain a homomorphism

$$G \to U(m) = \{ a \in \mathbb{C}^{m \times m} \mid a^* a = 1\!\!1 \}$$

$$(a^*)_{ij} = \overline{a_{ji}}$$

26. Theorem (Peter – Weyl Theorem)

[Hermann Weyl, 1885 – 1955, German math.
Fritz Peter 1889 – 1949 German math.]

Let G be a compact grp. Then there are integers $m_i > 0$, for $i \in I$ and an injective $\underbrace{\text{morphism}}_{\text{closed}}$

$$G \longrightarrow \prod_{i \in I} U(m_i).$$

Hence every compact grp is isomorphic to a closed subgrp of a product of unitary matrix groups $U(m_i)$.

proof Let $E \neq 0$ be a faithful Hilbert-G-module, e.g. $E = L^2(G, \mathbb{C})$ and decompose E as in §3.25, $E = \overline{\sum_{i \in I} F_i}$, F_i simple, $F_i \perp F_j$ for $i \neq j$. Put $m_i = \dim(F_i)$ so $F_i \cong \mathbb{C}^{m_i}$.

We obtain morphisms $\rho_i : G \to U(m_i)$ from the restriction map $g \mapsto g|_{F_i}$.

Since the module is faithful, $\ker(\rho) = \{e\}$.

Since G is compact, ρ is closed. $\square$

<u>Corollary</u> Let G be a compact group, let $g \in G$, $g \neq e$. Then there is a morphism $\rho: G \to U(m)$ for some $m > 0$, with $\rho(g) \neq \mathbb{1}$.

(cp. van Dantzig § 2.7 and § 2.8)

For abelian groups we can strengthen this.

27. <u>Lemma</u> Let H be an abelian group, let $E \neq \{0\}$ be a finite-dimensional complex simple H module (i.e. $E, \{0\}$ are the only H-invariant subspaces). Then $\dim(E) = 1$, $E \cong \mathbb{C}$.

<u>Proof</u> Let $h \in H$, let z be an eigenvalue of $h: u \mapsto hu$ with eigenspace E_z. For $g \in H$, $u \in E_z$ we have

$$\underset{\substack{\| \\ hgu}}{ghu} = gzu = zgu$$

$\Rightarrow E_z$ is H-invariant $\Rightarrow E_z = E$, $h(u) = zu$.

Thus every subspace of E is H-invariant

$\Rightarrow \dim E = 1$ $\square$

<u>Corollary</u> If G is a compact abelian group, then G is isomorphic to a closed subgrp of $\prod_{i\in I} U(1) = U(1)^I$ for some index set I, $U(1) = \{ z \in \mathbb{C} \mid |z| = 1 \}$. $\square$

28. <u>Theorem</u> (Approximation by matrix grps) Let G be a compact grp, let $V \subseteq G$ be an identity neighborhood. Then there is a closed normal subgrp $N \trianglelefteq G$ with $N \subseteq V$ such that $G/_N$ is isomorphic to a closed subgrp of $U(m)$ for some $m > 0$. #

<u>Proof</u> Fix a closed injective morphism $\rho : G \to \prod_{i\in I} U(m_i)$.

There is a finite set $I_0 \subseteq I$, open identity neighborhoods

$V_i \subseteq U(m_i)$ for $i \in I_0$

$V_j = U(m_j)$ for $j \in I - I_0$

such that $\rho(V) \subseteq \prod_{i\in I} V_i$. Put $K = \prod_{i\in I} K_i \subseteq \prod_{i\in I} V_i$

$K_i = \{1\}$ if $i \in I_0$

$K_j = U(m_j)$ if $j \in I - I_0$

$N = \rho^{-1}(K) \Rightarrow N \subseteq V$

$$\prod_{i\in I} U(m_i) \Big/ K \cong \prod_{i\in I_0} U(m_i) \subseteq U(m)$$

$$m = \sum_{i\in I_0} m_i$$

Then
$$\begin{tikzcd} G \arrow[r, "\rho"] \arrow[dr, "\tau"'] & \prod_{i\in I} U(m_i) \arrow[d] \\ & U(m) \end{tikzcd}$$

$\ker(\tau) = N \subseteq V$. $\square$

<u>Corollary</u> (No small subgrps criterion)
Let G be a compact grp. If there is an identity neighborhood $U \subseteq G$ such that U contains <u>no</u> closed normal subgroups $N \neq \{e\}$, then G is isomorphic to a closed subgrp of $U(m)$ for some $m > 0$, i.e. G is a <u>compact Lie group</u>.

The Hilbert-G-module $L^2(G, \mathbb{C})$ is special in the sense that it contains <u>all</u> simple G-modules.

29. <u>Theorem</u> Let G be a compact grp, let E be a simple Hilbert-G-module. Then $L^2(G,\mathbb{C})$ contains a submodule isomorphic to E (as a Hilbert-G-module).

<u>proof</u> Let $e_1,\dots,e_m$ be an orthonormal basis for E.(*) Put $f_k(x) = \langle xe_1 | e_k\rangle$.

<u>Claim:</u> the linear map h that maps e_k to f_k is G-equivariant.

<u>proof</u> Let $g \in G$. $ge_k = \sum_{j=1}^{m} \langle e_j | ge_k\rangle e_j$ (ONB)

$$(gf_k)(x) = f_k(g^{-1}x) = \langle g^{-1}xe_1 | e_k\rangle = \langle xe_1 | ge_k\rangle$$

$$= \sum_{j=1}^{m} \langle xe_1 | e_j\rangle\langle e_j | ge_k\rangle, \text{ hence } h(ge_k) = gf_k \ \square$$

There is a unique selfadjoint map $T: E \to h(E)$ such that $\langle Tu|v\rangle = \langle h(u) | h(v)\rangle$ for all $u,v\in E$.

For $a \in G$ we have

$$\langle Tau | av\rangle = \langle h(au) | h(av)\rangle$$
$$= \langle ah(u) | ah(v)\rangle = \langle h(u) | h(v)\rangle = \langle Tu|v\rangle$$

$\leadsto$ $a^{-1}Ta = T$ $\leadsto$ Every eigenspace of T is

---

(*) $\dim(E) < \infty$ by §4.22

G-invariant $\overset{E \text{ simple}}{\Downarrow}$ $T = s \cdot \mathrm{id}_E$ for some $s \neq 0$

$\rightsquigarrow s > 0$. Then $\sqrt{s} \cdot h$ is an isomorphism

$$\sqrt{s} \cdot h : E \longrightarrow F = \mathrm{span}\{f_1, \dots, f_m\} \subseteq C_c(G, \mathbb{C})$$

$\square$

<u>Final remark</u> Let G be a locally compact group. The <u>unitary dual</u> $\hat{G}$ is a set (of representatives) of all simple Hilbert-G-modules.

We have shown: if G is compact, then

$$L^2(G, \mathbb{C}) \cong \bigoplus_{E \in \hat{G}} \underbrace{(E \oplus \dots \oplus E)}_{m_E \geq 1}$$

A closer analysis shows that $m_E = \dim E$.

For locally compact groups the situation is <u>much</u> more complicated and the topic of ongoing research.