§2 Locally compact groups and the Haar integral

A Hausdorff topological grp G is called a <u>locally compact grp</u> if G carries a locally compact topology. Equivalently, $e \in G$ has some compact neighborhood.

If G is locally compact and if $H \subseteq G$ is closed, then G/H is locally compact (Exercise). (*)

1. <u>Lemma</u> Let G be a locally compact grp. If $U \subseteq G$ is a compact and open identity neighborhood, then U contains an open subgrp H.

<u>proof</u> We apply Wallace' Lemma §1.10 to $U \times \{e\} \subseteq U \times U$: there is an open symmetric identity neighborhood $V \subseteq U$ such that $UV \subseteq U$ (because $Ue = U \subseteq U$). Hence $VV \subseteq U$, and then $V.V.V \subseteq U$ etc so $\underbrace{V \cdots V}_{m} \subseteq U$ for all $m \geq 1$

so $H = \langle V \rangle \subseteq U$, and H is open by §1.9. $\square$

Examples of locally compact groups:

- every compact group, e.g. $(\mathbb{Z}/2)^{\mathbb{N}}$.
- $(\mathbb{R}^n, +)$, $(\mathbb{R}^*, \cdot)$, $(\mathbb{C}^*, \cdot)$
- $GL_n(\mathbb{R}) \subseteq \mathbb{R}^{n \times n}$ is open ($g \in GL_n(\mathbb{R}) \Leftrightarrow \det(g) \neq 0$) hence locally compact.
- every closed subgroup of a locally compact group is again locally compact.

2. Lemma Every locally compact group G has a σ-compact open subgroup $H \subseteq G$. In particular, every connected locally compact group is σ-compact.

proof Let $U \subseteq G$ be an open symmetric identity neighborhood. Then

$$\langle U \rangle = U \cup UU \cup UUU \cup \underbrace{U \cdots U}_{4} \cup \cdots$$

is σ-compact and open. $\square$

3. Def Let X be a topological space. The quasi-component of $a \in X$ is

$$Q(a) = \bigcap \{ D \subseteq X \mid a \in D,\ D \subseteq X \text{ closed and open} \}$$

Lemma If X is a compact space and $a \in X$, then $Q(a)$ is connected.

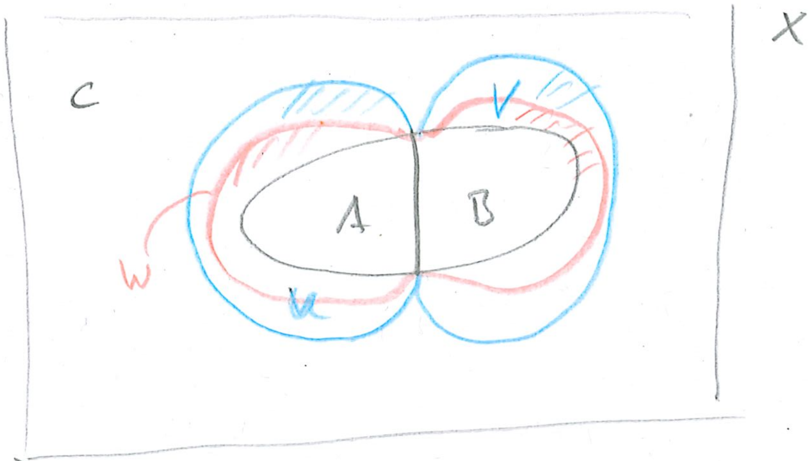
proof $Q(a)$ is closed and contains a.

Suppose that $Q(a) = A \cup B$, A, B closed, $A \cap B = \emptyset$, $a \in A$.

Claim: $B = \emptyset$

Since X is normal ($=T_4$), there are open sets $U, V \subseteq X$ with $A \subseteq U$, $B \subseteq V$, $U \cap V = \emptyset$

Put $G = X - (U \cup V) \Rightarrow G \cap Q(a) = \emptyset$



For every $c \in G$, there is a clopen set W_c with $a \in W_c$, $c \notin W_c$

Thus $G \subseteq \bigcup_{c \in G} \underbrace{(X - W_c)}_{\text{open}}$

$\Rightarrow G \subseteq (X - W_{c_1}) \cup \dots \cup (X - W_{c_m})$, because G is compact. Put $W = W_{c_1} \cap \dots \cap W_{c_m} \Rightarrow G \cap W = \emptyset$

whence $W \subseteq U \cup V \Rightarrow Q(a) \subseteq W$, since $a \in W$ and W clopen. Now $Y = U \cup (X - W)$ is open

$Z = V \cap W$ is open

$X = Y \cup Z$ hence Y is clopen. Thus

$Q(a) \subseteq Y \Rightarrow B = \emptyset$ $\square$

For the connected component $C(a)$ we have always $C(a) \subseteq Q(a)$. The previous Lemma says that $C(a) = Q(a)$ if X is compact. #

4. <u>Def</u> A Hausdorff space $X \neq \phi$ is called <u>zero-dimensional</u> if every $a \in X$ has arbitrarily small clopen neighborhoods: $W \subseteq X$ nbhd of $a \Rightarrow \exists$ clopen nbhd V of a with $V \subseteq W$.

Note: zero-dimensional $\Rightarrow$ totally disconnected.

<u>Lemma</u> A locally compact space X is zero-dimensional if and only if it is totally disconnected.

proof Assume X is locally compact and totally disconnected. Let $a \in X$, let W be a neighborhood of a. Replacing W by a smaller neighborhood, we may assume that W is open and that $\overline{W} \subseteq X$ is compact. <u>Claim</u>: there is a clopen neighborhood $V \subseteq X$ of a, with $V \subseteq W$.

If $\overline{W} = W$ put $V = W$ ~ done.

Otherwise, consider $\partial W = \overline{W} - W$ ~ $\partial W \neq \phi$ is compact.

Since $\overline{W}$ is compact and totally disconnected, $Q(a) = \{a\}$. Hence for every $b \in \partial W$, there is a compact neighborhood $U_b \subseteq \overline{W}$ of a with $b \notin U_b$, and U_b is open in $\overline{W}$. Then $\bigcap\limits_{b \in \partial W} U_b \cap \partial W = \emptyset$,

hence there are finitely many $b_1, \dots, b_m \in \partial W$ with $U_{b_1} \cap \dots \cap U_{b_m} \cap \partial W = \emptyset$. Put $U = U_{b_1} \cap \dots \cap U_{b_m}$

$\rightsquigarrow$ U compact, open in $\overline{W}$, $U \subseteq W$ $\rightsquigarrow$ U clopen in X. $\square$

5. <u>Theorem</u> (van Dantzig)

⌈ D. van Dantzig, Dutch math., 1900–1959 ⌋

Let G be a locally compact group. The following are equivalent: (i) G is totally disconnected

(ii) G is zero-dimensional

(iii) every identity neighborhood contains a compact open subgroup.

(iv) every identity neighborhood contains an open subgroup.

<u>proof</u> We have (i) $\overset{\S 2.4}{\Longleftrightarrow}$ (ii), (iii) $\Rightarrow$ (iv)

(ii) $\Rightarrow$ (iii): If $W \subseteq G$ is any identity nbhd, then W contains a compact open identity nbhd U by §2.4.

By §2.1, U contains a compact open subgp. $\square$

(iv) $\Rightarrow$ (i): Suppose $g \in G$, $g \neq e$. There is a clopen subgrp $H \subseteq G - \{g\}$ $\rightsquigarrow$ $g \notin G^0$ $\rightsquigarrow$ $G^0 = \{e\}$ $\square$

Theorem §2.5 shows in particular that in a locally compact totally disconnected group G, the compact open subgroups form a neighborhood basis of the identity.

6. Proposition Let G be a locally compact group, $H \subseteq G$ a closed subgroup. Then $G/_H$ is totally disconnected if and only if $H \subseteq G^0$.

proof. We consider the canonical map $q: G \to G/H$, $g \mapsto gH$. If $G^0 \not\subseteq H$, then $q(G^0)$ is a connected set containing more than 1 element $\Rightarrow$ $G/_H$ not totally disconnected.

Suppose now $G^0 \subseteq H$. Consider the canonical map $p: G/_{G^0} \to G/_H$, $p(gG^0) = gH$. This map is continuous and open:

$$\begin{tikzcd} G \arrow[r, equal] \arrow[d, "q_0"', "\text{open}"] & G \arrow[d, "q", "\text{open}"'] \\ G/_{G^0} \arrow[r, "p"] & G/_H \end{tikzcd}$$

$W \subseteq G/_{G^0}$ open $\Rightarrow$

$$p(W) = \underbrace{q(\underbrace{q_0^{-1}(W)}_{\text{open}})}_{\text{open}}$$

Let $a \in G - H$. Then $q_0(a) \neq q_0(e)$, hence there is a compact open subgroup $K \subseteq G/_{G^0}$ (by §2.5 and §1.15) with $q_0(a) \notin K$. Then $p(K) \subseteq G/_H$

is clopen, with $q(a) \notin p(K)$. Hence the connected component of $q(e)$ is $\{q(e)\} = \{H\}$.

But G acts transitively on G/H, hence the connected component of every element of G/H is trivial. $\square$

For compact groups, Theorem 5 can be improved.

7. __Theorem__ (van Dantzig) Let G be a compact group. The following are equivalent.

(i) G is totally disconnected

(ii) G is zero-dimensional

(iii) every identity neighborhood $V \subseteq G$ contains a compact open normal subgroup $N \trianglelefteq G$.

In particular, the compact open normal subgroups form then a neighborhood basis for $e \in G$.

__Proof__ By Theorem 5 we have (i) $\Leftrightarrow$ (ii) $\Leftarrow$ (iii).

__Claim:__ (ii) $\Rightarrow$ (iii). Let $V \subseteq G$ be an identity neighborhood. By §2.5, V contains an open compact subgroup $H \subseteq G$. Then G/H is finite (because $G = \bigcup_{g \in G} \underbrace{gH}_{\text{open}} \leadsto G = H \cup g_1 H \cup \dots \cup g_m H$)

We have a homomorphism $f: G \to \mathrm{Sym}(G/H)$

$$g \mapsto f(g), \quad f(g)(aH) = gaH$$

and $\ker(F) = N = \bigcap \{ gHg^{-1} \mid g \in G \}$

has finite index in G $\rightsquigarrow$ N is open in G

(because N is closed and $G = N \cup h_1 N \cup \ldots \cup h_n G$)

$\square$

$N \subseteq H \subseteq V$

8. Def A compact group G satisfying the equivalent conditions in §2.7 is called a profinite group.

Proposition A topological group G is profinite if and only if G is isomorphic (as a topological grp) to a closed subgrp of a product

$$F = \prod_{i \in I} F_i$$

of finite groups F_i (with the discrete topology on the F_i and the product topology on F). #

proof F as above is a compact grp, and F is totally disconnected (if $\emptyset \neq C \subseteq F$ is connected, the $p_i(C) \subseteq F_i$ is connected $\rightsquigarrow$ $p_i(C) = \{c_i\} \subseteq F_i$ for all i $\rightsquigarrow$ $C = \{(c_i)_{i \in I}\}$ is a singleton). Hence every closed subgrp $G \subseteq F$ is compact and totally disconnected, i.e. profinite.

Conversely, suppose that G is compact and totally disconnected. Put $\mathcal{N} = \{ N \trianglelefteq G \mid N \subseteq G$ is open $\}$ and $F = \prod_{N \in \mathcal{N}} \underbrace{G/N}_{\text{finite}}$, $f : G \to F$,

$f(g) = (gN)_{N \in \mathcal{N}}$. Then f is a morphism (since $pr_N \circ f(g) = gN$ is continuous for all $N \in \mathcal{N}$) and $\ker(f) = \bigcap \mathcal{N} = \{e\}$ by §2.7.

Hence $f : G \to F$ is injective. Since G is compact, F maps G homeomorphically onto $f(G)$. $\square$

9. <u>Def</u> Let G be a group (no topology on G). We call G <u>residually finite</u> if the following holds: for every element $a \in G$, $a \neq e$, there is a finite group E (depending on a) and a homomorphism $f : G \to E$ with $f(a) \neq f(e)$

The result above says that every profinite group is (as an abstract group) residually finite.

But there are other examples.

- $(\mathbb{Z}, +)$ is residually finite: for $k \neq 0$, put $E = \mathbb{Z}/m$, $f(l) = l + m\mathbb{Z}$, $m = |k| + 1$.

Then $f(k) = k + m\mathbb{Z} \neq m\mathbb{Z}$ since $m \nmid k$ (k is not a multiple of m).

- Every finite group is residually finite.

10. Def Let G be a residually finite group, let $\mathcal{N}_G = \{N \trianglelefteq G \mid [G:N] < \infty\}$ denote the set of all normal subgroups of finite index in G. Thus $\bigcap \mathcal{N}_G = \{e\}$, because for every $g \in G - \{e\}$ there is some $N \in \mathcal{N}_G$ with $g \notin N$. Put $F = \prod_{N \in \mathcal{N}_G} G/N$ with the product topology (so F is profinite) and consider $f: G \to F$, $g \longmapsto (gN)_{N \in \mathcal{N}_G}$. We denote the closure of $f(G)$ by $\hat{G} = \overline{f(G)}$ and call $i: G \to \hat{G}$, $g \mapsto f(g)$ the profinite completion of G.

11. Proposition (The universal property of the profinite completion)
Let G be a residually finite group, let $i: G \to \hat{G}$ be its profinite completion. Let K be any profinite group and let $h: G \to K$ be a group homomorphism.

Then there is a unique morphism $\hat{h} : \hat{G} \to K$ such that $h = \hat{h} \circ i$

$$\begin{tikzcd} G \arrow[r, "i"] \arrow[dr, "h"'] & \hat{G} \arrow[d, "\hat{h}\ \exists!"] \\ & K \end{tikzcd}$$

<u>proof</u> <u>Existence of $\hat{h}$.</u> Put $\mathcal{N}_G = \{ N \trianglelefteq G \mid [G:N] < \infty \}$

$\mathcal{N}_K = \{ M \trianglelefteq K \mid M \leq K \text{ open} \}$. For each $M \in \mathcal{N}_K$,

consider $M' = \ker(h \circ p_M)$ $\quad G \xrightarrow{h} K \xrightarrow{p_M} K/M$

Since K/M is finite, $M' \in \mathcal{N}_G$. We define a morphism

$$\check{h} : \prod_{N \in \mathcal{N}_G} G/N \longrightarrow \prod_{M \in \mathcal{N}_K} K/M$$

via $\quad (g_N N)_{N \in \mathcal{N}_G} \longmapsto \left(h_M (g_{M'} M') \right)_{M \in \mathcal{N}_K}$,

$$\begin{tikzcd} G \arrow[r, "h"] \arrow[d, "p_{M'}"'] & K \arrow[d, "p_M"] \\ G/M' \arrow[r, "h_M"] & K/M \end{tikzcd} .$$

Consider the commutative diagram

$f(k) = (kM)_{M \in \mathcal{N}_K}$

$$\begin{tikzcd} G \arrow[r, "i"] \arrow[d, "h"'] & \hat{G} \arrow[r, hook] \arrow[dl, "\hat{h}"'] \arrow[d, "\check{h}|_{\hat{G}}^{f(K)}"] & \prod_{N \in \mathcal{N}_G} G/N \arrow[d, "\check{h}"] \\ K \arrow[r, "f", "\cong"'] & f(K) \arrow[r, hook] & \prod_{M \in \mathcal{N}_K} K/M \end{tikzcd}$$

$\square$

Uniqueness of $\hat{h}$. Since $i(G) \subseteq \hat{G}$ is dense, $\hat{h}$ is uniquely determined by h,

$$\begin{tikzcd} G \arrow[r] \arrow[d, "h"'] & i(G) \subseteq \overline{i(G)} = \hat{G} \arrow[dl, bend left, "\hat{h}"] \\ K & \end{tikzcd}$$

$\square$

We make a useful observation.

Lemma Let G be a profinite group, let $H \subseteq G$ be a subgroup. Then $\overline{H} = \bigcap \{ K \mid K \subseteq G$ open subgrp with $H \subseteq K \}$.

proof Suppose $g \in G - \overline{H}$. Then there is an identity neighborhood V such that $gV \cap H = \emptyset$. Let $L \trianglelefteq G$ be an open normal subgrp with $L \subseteq V^{-1}$ and consider $K = HL$. This is a subgrp (because $L \trianglelefteq G$) and open (because L is open). Since $g \notin HV^{-1}$, $g \notin K$.

$\square$

12. Example The profinite completion of $\mathbb{Z}$

By definition, $\hat{\mathbb{Z}}$ is the closure of $f(\mathbb{Z}) \subseteq \prod_{m=2}^{\infty} \mathbb{Z}/m$, where $f(k) = (k + m\mathbb{Z})_{m=2}^{\infty}$. This is not helpful.

Fact: Let $m \geq 2$ be an integer, with prime decomposition

$$m = p_1^{n_1} \cdots p_\ell^{n_\ell}, \quad p_1 < \dots < p_\ell \text{ primes}, \; n_j \geq 1.$$

Then $\mathbb{Z}/m \cong \mathbb{Z}/p^{n_1} \times \dots \times \mathbb{Z}/p^{n_\ell}$

The proof uses for example the Chinese Remainder Theorem or Sylow's Theorem.

We therefore consider first $f_p : \mathbb{Z} \to \prod_{n=1}^{\infty} \mathbb{Z}/p^n$, $p \in \mathbb{P}$

$f_p(k) = (k + p^n\mathbb{Z})_{n=1}^{\infty}$ and $\mathbb{Z}_p = \overline{f_p(\mathbb{Z})} \subseteq \prod_{n=1}^{\infty} \mathbb{Z}/p^n$.

The elements of $\mathbb{Z}_p$ are called the p-adic integers.

Small observation: $\mathbb{Z}$ and $\mathbb{Z}/m$ are commutative rings. Hence $\hat{\mathbb{Z}}$ and $\mathbb{Z}_p$ are compact commutative rings.

Since $\ker f_p = \bigcap_{n=1}^{\infty} p^n\mathbb{Z} = \{0\}$, f_p is injective.

We view $\mathbb{Z}$ as a subring of $\mathbb{Z}_p$.

Some properties of $\mathbb{Z}_p$

For any abelian grp $(A,+)$ and $m \in \mathbb{Z}$, we have the endomorphism $a \mapsto ma = \underbrace{a+a+\dots+a}_{m}$ of A.

Lemma A If $\gcd(m,p)=1$, then the map $u \mapsto mu$ is a surjective endomorphism of $(\mathbb{Z}_p,+)$.

proof We claim that $\overline{m\mathbb{Z}} = \mathbb{Z}_p$. It suffices to show that $1 \in \overline{m\mathbb{Z}}$, for then $\mathbb{Z}_p = \overline{\mathbb{Z}} \subseteq \overline{m\mathbb{Z}}$.

Given $n \geq 1$, we have $\gcd(m,p^n)=1$, hence there are integers $a,b \in \mathbb{Z}$ with $am + bp^n = 1$ (Bézout's Lemma), whence
$$am + p^k\mathbb{Z} = 1 + p^k\mathbb{Z} \quad \text{for all } k \leq n.$$
Hence $1 \in \overline{m\mathbb{Z}}$. $\square$

Lemma B For $n \geq 1$, $p^n\mathbb{Z}_p \subseteq \mathbb{Z}_p$ is an open subgrp of index p^n.

proof Consider

$$\begin{tikzcd}
\mathbb{Z}_p \arrow[r, hook] \arrow[dr, "h"] & \prod_{k=1}^{\infty} \mathbb{Z}/p^k \arrow[d, "pr_n"] \\
\mathbb{Z} \arrow[u, hook] \arrow[r] & \mathbb{Z}/p^n
\end{tikzcd}$$

We have $p^n \prod_{k=1}^{\infty} \mathbb{Z}/p^k \subseteq \ker(pr_k)$, whence $p^n\mathbb{Z}_p \subseteq \ker(h)$.

Since $\ker(h) \subseteq \mathbb{Z}_p$ is open, $p^n\mathbb{Z} = \mathbb{Z} \cap \ker(h)$ is dense in $\ker(h)$.

Hence $\overline{p^n\mathbb{Z}} = \ker(h) \supseteq p^n\overline{\mathbb{Z}} = \underbrace{p^n\mathbb{Z}_p}_{\text{compact}} \supseteq \overline{p^n\mathbb{Z}} \Rightarrow \ker(h) = p^n\mathbb{Z}_p$.

$\square$

<u>Lemma C</u> Every closed subgrp $0 \neq H \subseteq \mathbb{Z}_p$ is of the form $H = p^n \mathbb{Z}_p$, some $n \geq 0$.

<u>proof</u> Let $L \subseteq \mathbb{Z}_p$ be an <u>open</u> subgrp, $m = [\mathbb{Z}_p : L]$.

If $q \in \mathbb{P}$, $q \neq p$, then $q\mathbb{Z}_p = \mathbb{Z}_p$ by Lemma A,

hence $q \frac{\mathbb{Z}_p}{L} = \frac{\mathbb{Z}_p}{L} \Rightarrow q \nmid m$ (Lagrange's Theorem).

Thus $m = p^n$ for some $n \geq 1$ and $p^n \mathbb{Z}_p \subseteq L$.

But $[\mathbb{Z}_p : p^n \mathbb{Z}_p] = [\mathbb{Z}_p : L]$, hence $L = p^n \mathbb{Z}_p$.

So every open subgroup $L \subseteq \mathbb{Z}_p$ is of the form $L = p^n \mathbb{Z}_p$.

But then every closed subgrp $\neq 0$ also of this form, since it is an intersection of open subgrps by §2.11 ⊛

$\square$

<u>Lemma D</u> $u \mapsto pu$ is injective on $\mathbb{Z}_p$.

<u>proof</u> Let $u \in \prod_{k=1}^{\infty} \mathbb{Z}/p^k$, $u = (u_k + p^k \mathbb{Z})$.

If $pu = 0$, then $u_k = p^{k-1} l_k$.

If $u \in \overline{\mathbb{Z}}$, then for each k there is some $m \in \mathbb{Z}$

with $m = u_k + p^k r_k = p^{k-1}(l_k + p r_k)$

$m = u_{k+1} + p^{k+1} r_{k+1} = p^k (l_{k+1} + p r_k)$

$\Rightarrow m \in p^k \mathbb{Z} \Rightarrow u = 0$.

$\square$

(*) In more detail: Let $H \subseteq \mathbb{Z}_p$ be a closed sub grp. If $H \neq 0$, then there is $u \in H$, $u \neq 0$. For $n \gg 1$, $u \notin p^n \mathbb{Z}_p$ since the $\{p^n \mathbb{Z}_p \mid n \geq 1\}$ form a neighbourhood basis of $\{0\}$. Hence $\{n \in \mathbb{N} \mid u \in p^n \mathbb{Z}\}$ is finite $\Rightarrow \bigcap_n \{p^n \mathbb{Z}_p \mid H \subseteq p^n \mathbb{Z}_p\}$ $= p^m \mathbb{Z}_p$ for some $m \geq 0$. ☐

Lemma E $\mathbb{Z}_p$ is an integral domain and contains no elements of (additively) finite order $\neq 0$.

proof Let $u \in \mathbb{Z}_p$ and put $A = \{a \in \mathbb{Z}_p \mid au = 0\}$. Then A is a closed subgroup of $(\mathbb{Z}_p, +)$, whence $A = 0$ or $A = p^n \mathbb{Z}_p$ for some $n \geq 0$. In particular, $p^n \mathbb{Z} \subseteq A$ if $A \neq 0$. If $u \neq 0$, then $A = 0$ by Lemma D. Since $\mathbb{Z} \subseteq A$, this implies $mu \neq 0$ if $u \neq 0$, $m \in \mathbb{Z}$, $m \neq 0$. $\square$

Lemma F Put $R = \prod_{p \in \mathbb{P}} \mathbb{Z}_p$, let $m \in \mathbb{N}$, $m \geq 2$.

Then $mR \subseteq R$ is the unique open subgroup of index m.

proof Put $m = p_1^{n_1} \cdots p_e^{n_e}$, $p_1 < p_2 < \dots < p_e$ primes, $n_1, \dots, n_e \geq 1$. Then $mR = \prod_{p \in \mathbb{P}} m\mathbb{Z}_p$

$$= p_1^{n_1}\mathbb{Z}_{p_1} \times \dots \times p_e^{n_e}\mathbb{Z}_{p_e} \times \prod_{\substack{q \in \mathbb{P} \\ q \neq p_1, \dots, p_e}} \mathbb{Z}_q$$

whence $R/_{mR} = \mathbb{Z}/p_1^{n_1} \times \dots \times \mathbb{Z}/p_e^{n_e} \cong \mathbb{Z}/m$, i.e. $[R : mR] = m$.

If $L \subseteq R$ is an open subgroup, $[R : L] = m$, then $mR \subseteq L$, whence $mR = L$. $\square$

Theorem The profinite completion of $\mathbb{Z}$ is

$$\hat{\mathbb{Z}} = \prod_{p \in \mathbb{P}} \mathbb{Z}_p.$$

proof Let $m \in \mathbb{N}$, $m \geq 2$. Put $f(k) = (f_p(k))_{p \in \mathbb{P}} \in R = \prod_{p \in \mathbb{P}} \mathbb{Z}_p$, $f_p : \mathbb{Z} \hookrightarrow \mathbb{Z}_p$. We have a commutative diagram

$$\begin{tikzcd}
m\mathbb{Z} \arrow[r, hook] \arrow[d, hook] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z}/m \arrow[d, "\cong"', "h_m"] \\
mR \arrow[r, hook] & R \arrow[r] & R/mR
\end{tikzcd}$$

from which we obtain $h : R \to \hat{\mathbb{Z}} \subseteq \prod_{m=2}^{\infty} R/mR$

$$h(u) = (u + mR)_{m=2}^{\infty}$$

and $$\mathbb{Z} \hookrightarrow R \hookrightarrow \hat{\mathbb{Z}} \subseteq \prod_{m=2}^{\infty} R/mR.$$

But R is compact and $\hat{\mathbb{Z}}$ is the closure of $\mathbb{Z}$, hence $R = \hat{\mathbb{Z}}$. □

HR

Now we turn to integration on locally compact groups. We fix some notation

13. Def Let X be a Hausdorff space. Then $C(X,\mathbb{R})$ denotes the vectorspace (and ring) of continuous real functions on X. The <u>support</u> of $f \in C(X,\mathbb{R})$ is $supp(f) = \overline{\{x \in X \mid f(x) \neq 0\}}$. We say that f has <u>compact support</u> if $supp(f)$ is compact. The continuous functions with compact support form a subspace $C_c(X,\mathbb{R}) \subseteq C(X,\mathbb{R})$.

Every $f \in C_c(X,\mathbb{R})$ is bounded and we put

$$\|f\|_\infty = \max\{|f(x)| \mid x \in X\} = \sup\{|f(x)| \mid x \in X\}$$

This is a <u>norm</u> on $C_c(X,\mathbb{R})$.

If $f(x) \leq h(x)$ holds for all x, we write $f \leq h$.

Put $C_c^+(X,\mathbb{R}) = \{f \in C_c(X,\mathbb{R}) \mid f \geq 0\}$. Every $f \in C_c(X,\mathbb{R})$ is of the form $f = f_+ - f_-$, with $f_+, f_- \in C_c(X,\mathbb{R})$ (put $f_+(x) = \max\{0, f(x)\}$)

14. Lemma A. Suppose that X is locally compact and that $C \subseteq X$ is compact. Then there is a continuous map $f: X \to [0,1]$ with compact support, such that $f(c) = 1$ for all $c \in C$.

Proof Each $c \in C$ has an open neighborhood with compact closure U_c. Since C is compact, $C \subseteq U_{c_1} \cup \dots \cup U_{c_m} = U$. Then $\overline{U} \subseteq \overline{U}_{c_1} \cup \dots \cup \overline{U}_{c_m}$ is compact. Since $\overline{U}$ is normal (T_4), (*) there is a continuous map $f: \overline{U} \to [0,1]$ with $f(c) = 1$ for all $c \in C$ and $f(x) = 0$ for all $x \in \partial U = \overline{U} - U$. We extend f to X by putting $f(z) = 0$ for $z \in X - U$. □

Lemma B Let G be a locally compact group, let $f \in C_c(G, \mathbb{R})$. For every $\varepsilon > 0$, there is a symmetric identity neighborhood $V \subseteq G$ such that

$|f(g) - f(h)| < \varepsilon$ holds for all $g, h \in G$ with

$g^{-1}h \in V$.

proof For every $a \in G$ we choose an identity neighborhood $W_a \subseteq G$ such that $|f(a) - f(a')| < \frac{\varepsilon}{2}$ holds for all $a' \in aW_a$ and an identity neighborhood U_a such that $U_a U_a \subseteq W_a$.

(*) This is Urysohn's Lemma.

Since $C = supp(f)$ is compact, there exist $a_1, \dots, a_m \in G$ with $C \subseteq a_1 U_{a_1} \cup \dots \cup a_m U_{a_m}$.
We choose a symmetric identity neighbourhood V such that $V^{-1} \cdot V \subseteq U_{a_1} \cap \dots \cap U_{a_m}$.

Suppose $g^{-1}h \in V$. If $gV \cap a_k U_{a_k} = \emptyset$ for all $k = 1, \dots, m$, then $f(g) = f(h) = 0$ (✓).
If $gV \cap a_k U_{a_k} \neq \emptyset$, then $g \in a_k U_{a_k} V^{-1}$, whence $gV \subseteq a_k U_{a_k} V^{-1} V \subseteq a_k W_k$. For $h \in gV$ we have then $|f(g) - f(h)| \leq |f(g) - f(a_k)| + |f(a_k) - f(h)|$
$< \varepsilon$. $\square$

In an entirely similar way, one shows that there is a symmetric identity neighbourhood $V \subseteq G$ such that $|f(g) - f(h)| < \varepsilon$ hold for all g, h with $hg^{-1} \in V$.

Now we recall a construction from algebra, the group ring.

15. Def The group ring. Let R be a commutative ring (e.g. $R=\mathbb{R}$ or $R=\mathbb{C}$) and let G be a group (without a topology.) Let $R[G]$ denote the free R-module with basis $\{g \in G\}$. The elements of $R[G]$ are thus formal finite linear combinations

$$\underline{r_1 g_1 + \dots + r_n g_n}, \quad r_1, \dots, r_n \in R, \quad g_1, \dots, g_n \in G.$$

$$= \sum_{g\in G} r_g \cdot g \qquad \underline{\text{almost all coefficients}} \ r_g = 0$$

The addition is the obvious one, $\sum_{g\in G} r_g g + \sum_{g\in G} s_g g$

$$= \sum_{g\in G} (r_g + s_g) g$$

and the multiplication in $R[G]$ is obtained by putting $(r_g \cdot g)(s_h \cdot h) = (r_g \cdot s_h)(gh)$.

Then $R[G]$ is a ring, non commutative if G is not abelian. The unit element is $1 \cdot e$, $1 \in R$, $e \in G$.

There is a surjective ring homomorphism

$$\varepsilon : R[G] \to R, \quad \sum_{g\in G} r_g \cdot g \mapsto \sum_{g\in G} r_g$$

the augmentation.

If M is an R-module (a "vector space over R") and if G acts (from the right) linearly on M (i.e. $m \mapsto mg$ is a linear map for each $g \in G$, and $(mg)h = m(gh)$ for all $g, h \in G$)

then M is a (right) $R[G]$-module if we put $m(r_g \cdot g) = (m r_g) g$

16. Def Let G be a locally compact group, let $f: G \to \mathbb{R}$ be a map. Recall that $\lambda_a(g) = ag$ for $a, g \in G$. If we put $fa = f \circ \lambda_a$, i.e. $(fa)(g) = f(ag)$, then G acts linearly (from the right) on $C_c(G, \mathbb{R})$ via

$$(fa)(g) = f(ag).$$

Hence $C_c(G, \mathbb{R})$ is a (right) $\mathbb{R}[G]$-module.

We note also that

$$\|f\|_\infty = \|fa\|_\infty$$

since
$$\|f\|_\infty = \sup\{\,|f(g)| \mid g \in G\,\} = \sup\{\,|f(ag)| \mid g \in G\,\}.$$

#

Let $I: C_c(G, \mathbb{R}) \to \mathbb{R}$ be a linear map. We call I a <u>left invariant integral</u> on the locally compact group if the following hold

(i) If $f \geq 0$, then $I(f) \geq 0$

(ii) If $a \in G$, then $I(f_a) = I(f)$

(iii) There is some $f \geq 0$ with $I(f) > 0$.

Note that (i) implies that $f \leq h \Rightarrow I(f) \leq I(h)$.

If I is a left invariant integral and if $s \in \mathbb{R}, s > 0$, then $s \cdot I$ is also a left invariant integral.

Our aim is to show the existence of a left invariant integral.

17. <u>Example</u> Let G be any group, with the discrete topology. Then G is locally compact and $C_c(G, \mathbb{R}) = \{ f: G \to \mathbb{R} \mid f(g) \neq 0$ only for finitely many $g \in G \}$

Hence we may define $I(f) = \underbrace{\sum_{g \in G} f(g)}_{\text{finite sum}}$

$\leadsto I$ is a left invariant integral on G.

18. We put $\mathbb{R}^+[G] = \{ \sum_{g\in G} r_g g \in \mathbb{R}[G] \mid r_g \geq 0$ for all $g \}$ and we note the following.
If $u, v \in \mathbb{R}^+[G]$, then $u \cdot v,\ u+v \in \mathbb{R}^+[G]$.

<u>Lemma</u> Let G be a locally compact group, and assume that $f, \alpha \in C_c^+(G, \mathbb{R})$. If $\alpha \neq 0$, then there is $u \in \mathbb{R}^+[G]$ with $f \leq \alpha u$.

<u>proof</u> Let $t = \|\alpha\|_\infty$, $s = \|f\|_\infty$ and
$U = \{ g \in G \mid \alpha(g) > \frac{t}{2} \}$. Then $U \neq \emptyset$ (since $t>0$).
Since $G' = \mathrm{supp}(f) \subseteq G$ is compact, there exist
$g_1, \dots, g_n \in G$ with $G' \subseteq g_1 U \cup \dots \cup g_n U$.
For $c \in g_k U$ we have $\alpha(g_k^{-1} c) > \frac{t}{2}$.
We put $u = \frac{2 \cdot s}{t}(g_1^{-1} + \dots + g_n^{-1}) \in \mathbb{R}^+[G]$
For $c \in G'$ we have the

$f(c) \leq s$ and $\alpha u(c) = \sum_{j=1}^{n} \frac{2 \cdot s}{t} \alpha(g_j^{-1} c) > s \cdot n$ $\square$

whence $f \leq \alpha u$.

19. Def Let G be a locally compact group, let $f, \alpha \in C_c^+(G, \mathbb{R})$, with $\alpha \neq 0$. We define

$$(f:\alpha) = \inf \{ \epsilon(u) \mid u \in \mathbb{R}[G]^+, f \leq \alpha u \}$$

$$= \inf \{ \sum_{g \in G} r_g \mid \sum_{g \in G} r_g g \in \mathbb{R}[G], r_g \geq 0, f(x) \leq \sum_{g \in G} r_g \alpha(gx) \}$$

By the previous Lemma § 2.18, this is a real number.

Lemma Let G be a locally compact group, let $f, h, \alpha, \beta \in C_c^+(G, \mathbb{R})$ with $\alpha, \beta \neq 0$. Then we have

(i) $(fa:\alpha) = (f:\alpha) = (f:\alpha a)$ for all $a \in G$

(ii) $(sf:\alpha) = s(f:\alpha)$ for all $s \in \mathbb{R}, s \geq 0$

(iii) $f \leq h \Rightarrow (f:\alpha) \leq (h:\alpha)$

(iv) $(f+h:\alpha) \leq (f:\alpha) + (h:\alpha)$

(v) $(f:\alpha) \leq (f:\beta)(\beta:\alpha)$

(vi) $|f|_\infty \leq (f:\alpha)|\alpha|_\infty$

Proof (i) – (iv) are clear from the definition.

(v): Suppose $u, v \in \mathbb{R}[G]^+$ with

$f \leq \beta u$ and $\beta \leq \alpha v$. Then $f \leq \alpha u v$

whence $(f:\alpha) \leq \epsilon(uv) = \epsilon(u)\epsilon(v)$

(vi) Suppose $u \in \mathbb{R}[G]^+$ with $f \leq \alpha u$.

Then $f(x) \leq \sum_{g \in G} r_g \alpha(gx) \leq \sum_{g \in G} r_g \|\alpha\|_\infty$ $\quad u = \sum_{g \in G} r_g g$

$= \epsilon(u) \|\alpha\|_\infty$, whence $\|f\|_\infty \leq \epsilon(u) \|\alpha\|_\infty$ $\square$

20. Lemma Let G be a locally compact group, let $f_1, f_2 \in C_c^+(G, \mathbb{R})$. Then there is $h \in C_c^+(G, \mathbb{R})$ such that for every $\varepsilon > 0$, there is an identity neighborhood $V \subseteq G$ such that for all $\alpha \in C_c^+(G, \mathbb{R})$, $\alpha \neq 0$, $\operatorname{supp}(\alpha) \subseteq V$ we have

$$(f_1 : \alpha) + (f_2 : \alpha) \leq (f_1 + f_2 : \alpha)(1 + \varepsilon) + \varepsilon(1 + \varepsilon)(h : \alpha)$$

proof By Lemma A § 2.14 there is a continuous $h : G \to [0,1]$ with compact support, such that $h(x) = 1$ for all $x \in \operatorname{supp}(f_1) \cup \operatorname{supp}(f_2)$. We put $\xi(x) = f_1(x) + f_2(x) + \varepsilon \cdot h(x)$ and we define $\tilde{f}_1, \tilde{f}_2$ as follows:

$\tilde{f}_1(x) = 0$ if $f_1(x) = 0$ and $\tilde{f}_1(x) = \dfrac{f_1(x)}{\xi(x)}$ if $x \in \operatorname{supp}(f_1)$. This is possible since $\xi(x) \neq 0$ for $x \in \operatorname{supp}(f_1)$.

Similarly,
$\tilde{f}_2(x) = 0$ if $f_2(x) = 0$ and $\tilde{f}_2(x) = \dfrac{f_2(x)}{\xi(x)}$ if $x \in \operatorname{supp}(f_2)$.

Then $\tilde{f}_1 + \tilde{f}_2 \le 1$. Choose $\delta > 0$ such that

$6 \cdot \|\xi\|_\infty \cdot \delta \le \varepsilon^3$ and $2\delta < \varepsilon^2$. By

Lemma B §2.14 there is a symmetric identity

neighbourhood $V \subseteq G$ such that

$$|f_1(x) - f_1(y)|, \ |f_2(x) - f_2(y)| < \delta \quad \text{for all } x^{-1}y \in V.$$

For $x^{-1}y \in V$ and $x, y \in \operatorname{supp}(f_1)$ that

$$|\tilde{f}_1(x) - \tilde{f}_1(y)| = \left| \frac{f_1(x)\xi(y) - f_1(y)\xi(x)}{\xi(x)\xi(y)} \right|$$

$$\le \frac{1}{\varepsilon^2} |f_1(x)\xi(y) - f_1(y)\xi(x)| \qquad (\text{since } \xi(x), \xi(y) \ge \varepsilon)$$

$$\le \frac{1}{\varepsilon^2} |f_1(x)\xi(y) - f_1(x)\xi(x)| + \frac{1}{\varepsilon^2} |f_1(x)\xi(x) - f_1(y)\xi(x)|$$

$$\le \frac{1}{\varepsilon^2} 2\delta \|f_1\|_\infty + \frac{1}{\varepsilon^2} \delta \|\xi\|_\infty \underset{f_1 \le \xi}{\le} \frac{3}{\varepsilon^2} \delta \|\xi\|_\infty \le \frac{\varepsilon}{2}$$

If $\tilde{f}_1(x) = 0 \ne \tilde{f}_1(y)$, then , $x^{-1}y \in V$, then

$$|\tilde{f}_1(x) - \tilde{f}_1(y)| = \left| \frac{f_1(y)}{\xi(y)} \right| \le \frac{1}{\varepsilon} \delta \le \frac{\varepsilon}{2}$$

Thus $|\tilde{f}_1(x) - \tilde{f}_1(y)| \le \frac{\varepsilon}{2}$ for all $x^{-1}y \in V$. #

Suppose $\alpha \in C_c^+(G, \mathbb{R})$, $\alpha \ne 0$, $\operatorname{supp}(\alpha) \subseteq V$.

Suppose that $u = \sum_{g\in G} r_g g \in \mathbb{R}^+[G]$ with $\xi \leq \alpha u$. Since $f_i = \tilde{f}_i \cdot \xi$, we have for $i=1,2$

$$f_i(x) \leq \sum_{g\in G} \tilde{f}_i(x)\, r_g\, \alpha(gx) \underset{\substack{\uparrow \\ \mathrm{Supp}(\alpha)\subseteq V}}{\leq} \sum_{g\in G} \left(\tilde{f}_i(g^{-1}) + \tfrac{\varepsilon}{2}\right) r_g \cdot \alpha(gx)$$

whence $(f_i:\alpha) \leq \sum_{g\in G} \left(\tilde{f}_i(g^{-1}) + \frac{\varepsilon}{2}\right) r_g$ and

$$(f_1:\alpha) + (f_2:\alpha) \leq \sum_{g\in G} \big(\underbrace{\tilde{f}_1(g^{-1}) + \tilde{f}_2(g^{-1})}_{\leq 1} + \varepsilon\big) r_g \leq (1+\varepsilon)\epsilon(u).$$

Hence
$$\begin{aligned}(f_1:\alpha) + (f_2:\alpha) &\leq (1+\varepsilon)(\xi:\alpha) \\ &= (1+\varepsilon)(f_1+f_2+\varepsilon h:\alpha) \\ &\leq (1+\varepsilon)(f_1+f_2:\alpha) + \varepsilon(1+\varepsilon)(h:\alpha)\end{aligned}$$

$\square$

21. Construction (A. Weil) Let G be a locally compact group, let $P = \{f \in C_c(G,\mathbb{R}) \mid f \geq 0, f \neq 0\}$. Since G is completely regular (§1.19), $P \neq \phi$. We fix $f_0 \in P$.

For $\alpha, f \in P$ we put $I(f,\alpha) = \dfrac{(f:\alpha)}{(f_0:\alpha)}$

(note that $\underbrace{\|f_0\|_\infty}_{\neq 0} \leq (f_0:\alpha)\|\alpha\|_\infty$, whence $(f_0:\alpha) \neq 0$)

By §2.19 we have $\dfrac{1}{(f_0:f)} \leq I(f,\alpha) \leq (f:f_0)$

$I(fa,\alpha) = I(f,\alpha)$ for all $a \in G$

$I(\rho f,\alpha) = \rho I(f,\alpha)$ for all $\rho > 0$

$I(f_1+f_2,\alpha) \leq I(f_1,\alpha) + I(f_2,\alpha)$ for all $f_1, f_2 \in P$.

We consider the infinite compact cube $Q = \prod_{f\in P} \left[\frac{1}{(f_0:f)}, (f:f_0)\right]$

If $V \subseteq G$ is a symmetric identity neighborhood, then we put $Q_V = \{ (I(f,\alpha))_{f\in P} \mid \operatorname{supp}(\alpha) \subseteq V \} \subseteq Q$

If $W \subseteq V$, then $Q_W \subseteq Q_V$, hence the set $\{Q_V \mid V \text{ identity neighborhood}\}$ has the finite intersection property. By compactness, there exists some

$I \in \bigcap \{ \overline{Q_V} \subseteq Q \mid V \subseteq G \text{ identity neighborhood} \}$

We may view I as a map $I: P \to \mathbb{R}$, $f \mapsto I(f)$.
$(I = (I(f))_{f \in P} \in Q)$

22. __Theorem__ (A. Haar) Let G be a locally compact group. Then there exists a left invariant integral I on G, as in § 2.16.

[A. Haar, 1885 – 1933, Hungarian math.]

__Proof__ We put f_0, P, I as in § 2.21.

For all $\alpha, f \in P$, $s > 0$, $a \in G$ we have

$$I(sf, \alpha) - sI(f, \alpha) = 0 \Rightarrow I(sf) - sI(f) = 0$$

$$I(f_a, \alpha) - I(f, \alpha) = 0 \Rightarrow I(f_a) - I(f) = 0$$

For $f_1, f_2 \in P$ we have

$$I(f_1, \alpha) + I(f_2, \alpha) - I(f_1 + f_2, \alpha) \geq 0$$

$$\Rightarrow I(f_1) + I(f_2) - I(f_1 + f_2) \geq 0$$

Let $h \in P$ as in § 2.20. For every $\varepsilon > 0$, there is an identity neighborhood $V_\varepsilon \subseteq G$ such that

$$(f_1 : \alpha) + (f_2 : \alpha) \leq (f_1 + f_2 : \alpha)(1 + \varepsilon) + \varepsilon(1 + \varepsilon)(h : \alpha)$$

hold whenever $\mathrm{supp}(\alpha) \subseteq V_\varepsilon$.

Hence

$$I(f_1, \alpha) + I(f_2, \alpha) \leq I(f_1 + f_2, \alpha)(1 + \varepsilon) + \varepsilon(1 + \varepsilon)\underbrace{(h : f_0)}_{\text{constant!}}$$

and thus $I(f_1)+I(f_2) \leq I(f_1+f_2)(1+\varepsilon) + \varepsilon(1+\varepsilon)(h:f_0)$

for all $\varepsilon > 0$. Hence $I(f_1)+I(f_2) = I(f_1+f_2)$.

Now we put $I(f) = 0$ if $f = 0$.

For $f_1, f_2 \in P$ we put

$I(f_1 - f_2) = I(f_1) - I(f_2)$. This is well-defined: if $f_1 - f_2 = f_1' - f_2'$ for $f_1', f_2' \in P$, then $f_1 + f_2' = f_1' + f_2$, whence $I(f_1) + I(f_2') = I(f_1') + I(f_2)$.

Then $I : C_c(G, \mathbb{R}) \to \mathbb{R}$ is linear. We have

$I(f_0, \alpha) = 1$, hence $I(f_0) = 1$. Therefore

I is a left invariant integral. $\square$

Our next aim is to show that I is essentially unique.

23 <u>Lemma</u> Let G be a locally compact grp, let J be any left invariant integral on G. Let $\alpha, f \in C_c^+(G, \mathbb{R})$, $\alpha \neq 0$. Then

$$J(f) \leq (f : \alpha)\, J(\alpha).$$

In particular, $J(\alpha) \neq 0$.

<u>proof</u> Let $u \in \mathbb{R}^+[G]$, $u = \sum_{g\in G} r_g g$ with $f \leq \alpha u$, as in § 2.18. We have thus

$$f \leq \sum_{g\in G} r_g(\alpha\circ\lambda_g), \text{ whence } J(f) \leq \sum_{g\in G} r_g J(\alpha)$$

$$= \epsilon(u)\cdot J(\alpha),$$

hence $J(f) \leq (f:\alpha)J(\alpha)$. Since there is some $f \in C_c^+(G,\mathbb{R})$ with $J(f)>0$, this implies $J(\alpha)>0$ □

24. <u>Example</u> Let $G=\mathbb{R}$. If $f\in C_c(\mathbb{R},\mathbb{R})$, then there is an interval $[u,v]\subseteq\mathbb{R}$ with $\operatorname{supp}(f) \subseteq [u,v]$. Put $I(f) = \int_u^v f(t)\,dt$. This is a left invariant integral: for $a\in\mathbb{R}$, then $\int_{u-a}^{v-a} f(t+a)\,dt$

$$= \int_u^v f(t)\,dt.$$

More generally, if $G=\mathbb{R}^m$ and if λ^m is the m-dimensional Lebesgue measure on $\mathbb{R}^m$, then

$$\int_{\mathbb{R}^m} f\,d\lambda^m = I(f)$$ is a left invariant integral.

25. Theorem (Haar) (Uniqueness of the left invariant integral)

Let G be a locally compact group, let I, J be two left invariant integrals on G. Then there is a constant $s > 0$ such that $J = s \cdot I$, i.e.

$$J(f) = s\, I(f) \quad \text{for all } f \in C_c(G, \mathbb{R}).$$

proof Let $P = \{ f \in C_c^+(G, \mathbb{R}) \mid f \neq 0 \}$ and fix $f_0 \in P$. It suffices to show that

$$\frac{I(f_1)}{I(f_0)} = \frac{J(f_1)}{J(f_0)} \quad \text{holds for all } f_1 \in P$$

(note: $I(f_0) \neq 0 \neq J(f_0)$ by § 2.23). For then

$$J(f_1) = \underbrace{\frac{J(f_0)}{I(f_0)}}_{s} I(f_1) \quad \text{holds for all } f_1 \in P$$

$$\rightsquigarrow J(f) = \frac{J(f_0)}{I(f_0)} I(f) \quad \text{for all } f \in C_c(G, \mathbb{R}).$$

#

The proof uses the following claim which we show afterwards

Claim ⊛ Given $f \in P$ and $\varepsilon > 0$, there is $\alpha \in P$ such that

$$(1-\varepsilon)(f : \alpha) J(\alpha) \leq J(f)$$

and

$$(1-\varepsilon)(f_0 : \alpha) J(\alpha) \leq J(f_0)$$

and similarly for I.

From this claim and $J(f) \le (f:\alpha) J(\alpha)$ we obtain $$(1-\varepsilon)\frac{(f:\alpha)}{(f_0:\alpha)} \le \frac{1}{(f_0:\alpha)}\frac{J(f)}{J(\alpha)} \le \frac{J(f)}{J(f_0)}$$

and $$(1-\varepsilon)\frac{(f_0:\alpha)}{(f:\alpha)} \le \frac{J(f_0)}{J(f)} \leadsto \frac{1}{1-\varepsilon}\frac{(f:\alpha)}{(f_0:\alpha)} \ge \frac{J(f)}{J(f_0)}$$

whence $$(1-\varepsilon)\frac{(f:\alpha)}{(f_0:\alpha)} \le \frac{J(f)}{J(f_0)} \le \frac{1}{1-\varepsilon}\frac{(f:\alpha)}{(f_0:\alpha)}$$

$$\Rightarrow \frac{I(f)}{I(f_0)} \le \frac{1}{1-\varepsilon}\frac{(f:\alpha)}{(f_0:\alpha)} \le \frac{1}{(1-\varepsilon)^2}\frac{J(f)}{J(f_0)} \quad \text{for all } \varepsilon > 0$$

$$\Rightarrow \frac{I(f)}{I(f_0)} \le \frac{J(f)}{J(f_0)}$$ similarly $$\frac{J(f)}{J(f_0)} \le \frac{I(f)}{I(f_0)}$$ so the result follows. □

We still need to prove Claim ⊛.

Proof of Claim ⊛ We put $f = f_1$, need to show that given $\varepsilon > 0$ there is $\alpha \in P$ such that
$$(1-\varepsilon)(f_j:\alpha) J(\alpha) \le J(f_j) \qquad j = 1, 0.$$
Put $C = \mathrm{supp}(f_0) \cup \mathrm{supp}(f_1)$, let $\eta : G \to [0,1]$, $\eta \in C_c(G, \mathbb{R})$ with $\eta(c) = 1$ for all $c \in C$, cp §2.14.
We choose $\delta > 0$ so that $\delta \cdot (\eta : f_j) < \frac{\varepsilon}{2}$, $j = 0, 1$.
Let $V \subseteq G$ be a symmetric identity neighborhood such that $|f_j(x) - f_j(y)| < \delta$ for $j = 0, 1$, $x^{-1}y \in V$.

Let $\beta \in P$ with $\operatorname{supp}(\beta) \subseteq V$, put $\alpha(x) = \beta(x) + \beta(x^{-1})$ $\leadsto$ $\operatorname{supp}(\alpha) \subseteq V$, $\alpha \in P$, $\alpha(x) = \alpha(x^{-1})$.

We claim this is the right α for Claim ⊛.

Note: these data do not depend on J or I.

Choose $t > 0$ such that $t \cdot J(f_j) \leq \delta J(\alpha)$ $j = 0, 1$.

Choose an open symm. identity neighborhood $W \subseteq G$ with $\overline{W}$ compact, such that $|\alpha(x) - \alpha(y)| < t$ for $x^{-1}y \in W$.

Since $C' \subseteq G$ is compact, there are $g_1, \dots, g_m \in G$ with $C' \subseteq g_1 W \cup \dots \cup g_m W$. Put $U_k = g_k W_k$ and $U_0 = G - C'$ $\leadsto$ $G = U_0 \cup \dots \cup U_m$.

Since G is locally compact, there is a partition of unity $\varphi_0, \dots, \varphi_m \in C_c(G, \mathbb{R})$, $\varphi_k : G \to [0,1]$, $\operatorname{supp}(\varphi_k) \subseteq U_k$, $\sum_{k=0}^{m} \varphi_k = 1$, cp Dieudonné VIII §4.2 or Engelking §5.1.9.

For $c \in C$ we have then $\sum_{k=1}^{m} \varphi_k(c) = 1$, and thus

$$\textcircled{1} \quad F_j(x) = \sum_{k=1}^{m} F_j(x) \cdot \varphi_k(x) \qquad \forall x \in G$$

For $x^{-1}y \in V$ we have $f_j(x) - s \leq f_j(y)$

$\rightsquigarrow$ $(f_j(x) - s) \cdot \alpha(x^{-1}y) \leq f_j(y) \cdot \alpha(x^{-1}y)$ $\quad \forall x, y \in G$

$\Rightarrow$ (2) $(f_j(x) - s)\, J(\alpha) \leq J(f_j \cdot (\alpha x^{-1}))$ $\quad \forall x \in G$

For $y \in U_k$, $g_h^{-1} y = g_h^{-1} x x^{-1} y \in W$, hence

$$\alpha(x^{-1}y) \leq \alpha(x^{-1} g_h) + t = \alpha(g_h^{-1} x) + t$$

$\rightsquigarrow$ $\varphi_h(y)\, \alpha(x^{-1}y) \leq \varphi_h(y)\, \alpha(g_h^{-1}x) + t \cdot \varphi_h(y)$ $\quad \forall y \in G$

$\rightsquigarrow$ $f_j(y) \cdot \varphi_h(y) \cdot \alpha(x^{-1}y) \leq f_j(y) \cdot \varphi_h(y) \cdot \alpha(g_h^{-1}x) + t \cdot f_j(y)\, \varphi_h(y)$

Hence

(3) $$J(f_j \cdot (\alpha x^{-1})) \leq \sum_{h=1}^{m} J(f_j \cdot \varphi_h)\, \alpha(g_h^{-1}x) + t\, J(f_j)$$

$$\leq \sum_{h=1}^{m} J(f_j \cdot \varphi_h)\, \alpha(g_h^{-1}x) + s\, J(\alpha)$$

$\Rightarrow$ $$(f_j(x) - s)\, J(\alpha) \leq \sum_{h=1}^{m} J(f_j \cdot \varphi_h)\, \alpha(g_h^{-1}x) + s \cdot J(\alpha)$$

Put $f_j' = \max\{0, f_j - 2 \cdot s\}$ $\rightsquigarrow$ $f_j' \geq 0$, $f_j' \in C_c(G, \mathbb{R})$

$\rightsquigarrow$ $$f_j'(x) \cdot J(\alpha) \leq \sum_{h=1}^{m} J(f_j \cdot \varphi_h)\, \alpha(g_h^{-1}x)$$

$\rightsquigarrow$ $$(f_j' : \alpha) \cdot J(\alpha) \leq \sum_{h=1}^{m} J(f_j \cdot \varphi_h) = J(f_j)$$

and $J(f_j) \leq (f_j : \alpha) J(\alpha)$.

Now $f_j \leq f_j' + 2s\eta$, hence

$$(f_j : \alpha) \leq (f_j' + 2s \cdot \eta : \alpha) \leq (f_j' : \alpha) + 2s(\eta : \alpha)$$

$$\leq (f_j' : \alpha) + 2s(\eta : f_j)(f_j : \alpha)$$

$$\leq (f_j' : \alpha) + \varepsilon (f_j : \alpha)$$

$$\Rightarrow (f_j : \alpha)(1 - \varepsilon) J(\alpha) \leq J(f_j)$$

$\square$

This left invariant integral I, which is unique up to a scalar $s > 0$, is called the <u>Haar integral</u> on the locally compact group G. We write

$$I(f) = \int_G f(g)\, dg .$$

If G is compact, then the function $f(g) = 1 = \text{const}$ has compact support. In this case we can <u>normalize</u> the Haar integral such that

$$\mathrm{vol}(G) = \int_G 1\, dg = \underline{1} .$$

26. Def Let G be a locally compact group, let $a \in G$. Consider for $f \in C_c(G, \mathbb{R})$

$$I_a(f) = \int_G f(ga^{-1})\,dg = \int_G (f \circ \rho_{a^{-1}})(g)\,dg = I(f \circ \rho_{a^{-1}})$$

where $\rho_x(y) = yx$.

Then I_a is a left invariant integral, since

$$\int_G f(bga^{-1})\,dg = \int_G (f \circ \rho_{a^{-1}} \circ \lambda_b)(g)\,dg =$$

$$\int_G (f \circ \rho_{a^{-1}})(g)\,dg.$$ By Thm § 2.25 there is $s > 0$ such that $I_a = sI$.

We put $\mathrm{mod}(a) = s$ and call $\mathrm{mod}(a)$ the modulus of a, and mod the modular function.

27 Theorem Let G be a locally compact group, let $a \in G$. Then $\mathrm{mod}(a)$ is independent of the chosen Haar integral I. The map $\mathrm{mod}: G \to \mathbb{R}^*$ is a morphism of topological groups.

<u>proof</u> Let J be another left invariant integral. Then $J = t\cdot I$ for some $t>0$, and

$$J(f\circ \varrho_{a^{-1}}) = t\,I(f\circ\varrho_{a^{-1}}) = t\cdot \mathrm{mod}(a)\,I(f) = \mathrm{mod}(a)\,J(f),$$

hence mod is independent of I.

For $a,b\in G$ we have $I(f\circ\varrho_{a^{-1}}\circ\varrho_{b^{-1}}) = \mathrm{mod}(b)\,I(f\circ\varrho_{a^{-1}})$

$= \mathrm{mod}(b)\,\mathrm{mod}(a)\,I(f) = I(f\circ\varrho_{b^{-1}a^{-1}}) = I(f\circ\varrho_{(ab)^{-1}})$

$= \mathrm{mod}(ab)\cdot I(f)$, hence mod is a group homomorphism.

In order to show that mod is continuous, it suffices to show continuity at $e\in G$, cp. Problem 1.1.

We choose $f\in P = \{h\in C_c(G,\mathbb{R}) \mid h\geq 0,\ h\neq 0\}$ and put $C = \mathrm{supp}(f)$. Let $U\subseteq G$ be an identity neighbourhood with $\overline{U}$ compact. Since $C\overline{U}\subseteq G$ is compact, there is $\eta: G\to[0,1]$, $\eta\in C_c(G,\mathbb{R})$ with $\eta(x)=1$ for all $x\in C\overline{U}$. Let now $\varepsilon>0$.

Let $V = V_\varepsilon \subseteq U$ be open and symmetric, such that

$$|f(x)-f(y)|\cdot I(\eta) < \varepsilon\cdot I(f) \quad \text{for } x^{-1}y\in V.$$

For $x\in G$, $a\in V$ we have

$|f(xa^{-1}) - f(x)| \cdot I(\eta) < \varepsilon \cdot \eta(x) I(f)$, because:

$$\begin{cases} x \in C\bar{U} \leadsto \eta(x) = 1 \quad \text{and} \quad x^{-1} x a^{-1} = a^{-1} \in V \\ x \notin C\bar{U} \leadsto x \notin C \text{ and } xa^{-1} \notin C \leadsto f(xa^{-1}) = 0 = f(x) \end{cases}$$

Hence $|I_a(f) - I(f)| \cdot I(\eta) \leq \varepsilon I(\eta) \cdot I(f)$

$\leadsto |\text{mod}(a) - 1| \leq \varepsilon$ for $a \in V$. $\square$

28. <u>Def</u> We call a locally compact group G <u>unimodular</u> if $\text{mod} : G \to \mathbb{R}^*$ is constant.

This holds if and only if the Haar integral is <u>right invariant</u>, i.e. iff $I(f \circ \rho_{a^{-1}}) = I(f)$ for all $a \in G$.

<u>Prop</u> The following types of groups are unimodular.

(i) abelian locally compact groups

(ii) compact groups

(iii) locally compact groups with $[G,G] = G$

(recall: $[G,G] = \langle [x,y] \mid x,y \in G \rangle$,

$[x,y] = xyx^{-1}y^{-1} = xy(yx)^{-1}$)

proof. (i) If G is abelian, $\rho_{a^{-1}} = \lambda_{a^{-1}}$.

(ii) If G is compact, then $\mathrm{mod}(G) \subseteq \mathbb{R}^*$ is compact. If $t \in \mathbb{R}^*$, $t \neq 1$, then $\langle t \rangle = \{ t^n \mid n \in \mathbb{Z} \}$ is unbounded. Hence $\mathrm{mod}(G) = \{1\}$.

(iii) $\mathbb{R}^*$ is abelian, hence $[G,G] \subseteq \ker(\mathrm{mod})$. □

29. Examples (a) $G = \mathbb{R}^m$

$$I(f) = \int_{\mathbb{R}^m} f \, d\lambda^m$$

← Lebesgue measure

is translation invariant, and $\mathbb{R}^m$ is unimodular (abelian).

(b) G a discrete grp, $f : G \to \mathbb{R}$ with compact = finite support

$$I(f) = \sum_{g \in G} f(g)$$ is left and right invariant

$$\sum_{g \in G} f(ag) = \sum_{g \in G} f(g) = \sum_{g \in G} f(ga^{-1})$$

Hence discrete grps are unimodular.

(c) $G = GL_m\mathbb{R} = \{ g \in \mathbb{R}^{m\times m} \mid \det(g) \neq 0 \}$

Note: G is open in $\mathbb{R}^{m\times m}$. Let $f \in C_c(G,\mathbb{R})$

Ansatz: $I(f) = \int_G f \cdot h \, d\lambda^{m\times m}$ ← Lebesgue measure

for some fixed $h: G \to \mathbb{R}$. We have for $a \in G$ the Transformation formula:

$$\int_G (f\circ\lambda_a)\cdot(h\circ\lambda_a)\,|\det(D\lambda_a)|\,d\lambda^{m\times m} =$$

$$\int_G f\cdot h \, d\lambda^{m\times m}$$

Note: $\lambda_a : \mathbb{R}^{m\times m} \to \mathbb{R}^{m\times m}$ is linear, hence $D\lambda_a = \lambda_a$ and $\det(\lambda_a) = \det(a)^m$.

Hence we want $h(xa)\,|\det(a)|^m = 1$, or

$h(x) = |\det(x)|^{-m}$

$\rightsquigarrow$ $I(f) = \int_G f(x)\cdot|\det(x)|^{-m}\, d\lambda^{m\times m}(x)$

This group is unimodular (→ Exercise)

(d) $G = \left\{ \begin{pmatrix} s & t \\ 0 & 1 \end{pmatrix} \,\middle|\, s,t \in \mathbb{R},\ s>0 \right\}$

is locally compact and <u>not</u> unimodular

(→ Exercises)

30 <u>Remark</u> For $x, g \in G$ put $\gamma_g(x) = g x g^{-1}$

Then $\gamma : G \to \mathrm{Aut}(G)$, $g \mapsto \gamma_g$

is a homomorphism with kernel $\mathrm{Cen}(G) =$

$\{ g \in G \mid gx = xg$ for all $x \in G \}$.

If G is locally compact, then

$$I(f \circ \gamma_a) = \int_G f(a g a^{-1})\, dg = \int_G f(g a^{-1})\, dg$$

$$= \mathrm{mod}(a)\, I(f)$$

Thus mod factors as

$$\begin{array}{ccc} G & \xrightarrow{\mathrm{mod}} & \mathbb{R}^* \\ {\scriptstyle q}\downarrow & \nearrow & \\ G/\mathrm{Cen}(G) & & \end{array}$$