§1 Topological groups

1. Def Let G be a group and let $\mathcal{T}$ be a topology on the set G. We do not require that the topology is Hausdorff (for reasons that will become clear soon).

We call $\mathcal{T}$ a group topology if the maps

$$m : G \times G \to G, \qquad (g,h) \longmapsto gh$$
$$i : G \to G \qquad g \longmapsto g^{-1}$$

are continuous.

Equivalently, the map $\varkappa : G \times G \to G$
$$(g,h) \longmapsto g^{-1}h$$

is continuous (because $i(g) = \varkappa(g,e)$ and $m(g,h) = \varkappa(i(g),h)$).

If $\mathcal{T}$ is a group topology, we call $(G, \cdot, \mathcal{T})$ a topological group.

We abbreviate this by saying "G is a topological group".

<u>Observations</u> For each $a \in G$, the maps

$\lambda_a : g \mapsto ga$, $\rho_a : g \mapsto ag$ and

$\gamma_a : g \mapsto aga^{-1}$ are homeomorphisms,

with inverses $\lambda_{a^{-1}}, \rho_{a^{-1}}, \gamma_{a^{-1}}$.

In particular, G is <u>homogenous</u> as a topological space: for all $a, b \in G$ there is a homeomorphism $f : G \to G$ with $f(a) = b$, eg $f(g) = ba^{-1}g = \rho_{ba^{-1}}(g)$ (or $f(g) = ga^{-1}b = \lambda_{a^{-1}b}(g)$.)

A <u>morphism</u> between topological groups K, G is a <u>continuous homomorphism</u>

$$f : G \to K$$

i.e. f is continuous <u>and</u> $f(gh) = f(g)f(h)$ for all $g, h \in G$.

2. <u>Examples</u> (a) Every group is a topological group for the trivial topology $\mathcal{T} = \{\emptyset, G\}$ (no open sets besides $\emptyset, G$) and for the discrete topology $\mathcal{T}_{disc} = \{U \subseteq G \mid U \text{ any subset}\}$.

(b) $(\mathbb{R},+)$ and $(\mathbb{R}^*,\cdot)$ are topological groups (with respect to the usual topology)

$f: \mathbb{R} \to \mathbb{R}^*$, $t \mapsto \exp(t)$ is a morphism, exp is continuous and $\exp(s+t) = \exp(s)\cdot\exp(t)$

(c) The <u>circle group</u> $\mathbb{S}^1 = U(1) = \{ z \in \mathbb{C} \mid |z|=1 \}$ is a topological group. The map $f: t \mapsto \exp(2\pi i t)$, $i = \sqrt{-1}$, is a morphism $\mathbb{R} \to U(1)$.

In coordinates: $\mathbb{S}^1 = \{ (x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 1 \}$

$f(t) = (\cos(2\pi t), \sin(2\pi t))$

(d) Suppose that $f: \mathbb{R} \to \mathbb{R}$ is a morphism. Then there is a <u>unique</u> $r \in \mathbb{R}$ such that $f(t) = r\cdot t$ holds for all $t \in \mathbb{R}$ (Exercise)

(e) Let $\mathbb{R}^\delta = \mathbb{R}$ with the discrete topology. Then $\mathbb{R}^\delta$ is a topological grp and $f: \mathbb{R}^\delta \to \mathbb{R}$, $t \mapsto t$ is a <u>bijective morphism</u>, but f is not an isomorphism of topological groups, f^{-1} is not continuous.

(F) Let F be a <u>topological field</u>, i.e. addition and multiplication are continuous, $(F^*,\cdot)$ is a topological grp, $(F,+)$ is a topological grp.

E.g. $F=\mathbb{R}$, $F=\mathbb{Q}$, $F=\mathbb{C}$.

Let $GL_m(F)=\{g\in F^{m\times m} \mid \det(g)\neq 0\}\subseteq F^{m\times m}$ with the subspace topology. Then $GL_m(F)$ is a topological grp.

<u>Proof</u> For $g=(g_{ij})_{i,j=1}^{m} \in F^{m\times m}$ put

$$g^\# = (a_{ij})_{i,j=1}^{m} \qquad a_{ij} = (-1)^{i+j}\det\left(\begin{array}{c|c} \times & \times \\ \hline \times & \times \end{array}\right)_{j}$$

$$i$$

remove ith column, jth row from g

Then $g g^\# = g^\# g = \det(g)\,\mathbb{1}$ (linear algebra...)

whence $g^{-1} = \frac{1}{\det(g)}\, g^\# = i(g)$ if g is invertible, <u>continuous</u>

$m(g,h) = (b_{ij})_{ij} \qquad b_{ij} = \sum_k g_{ik} h_{kj}$ <u>continuous</u>

$\square$

3. <u>Lemma</u> Let G be a topological grp. The following are equivalent: (i) the group topology is Hausdorff

(ii) there is some $a \in G$ such that $\{a\} \subseteq G$ is closed.

<u>Proof.</u> Clearly, (i) $\Rightarrow$ (ii) because points or finite sets in Hausdorff spaces are closed.

(ii) $\Rightarrow$ (i): Put $f(g,h) = g^{-1}ha$, $f: G \times G \to G$. Then $f^{-1}(a) = \{(g,h) \in G \times G \mid g^{-1}h = e\} = \{(g,g) \mid g \in G\}$ is the <u>diagonal</u> in $G \times G$. Hence G is Hausdorff if $\{a\} \subseteq G$ is closed. $\square$

In short: for topological grps, $T_1 \Rightarrow T_2$ (actually $T_1 \Rightarrow T_3$)

4. <u>Theorem</u> Let $I \neq \emptyset$ be an index set (finite or infinite) and let $(G_i)_{i \in I}$ be a family of topological grps. Put

$$G = \prod_{i \in I} G_i$$ with the product topology.

Then G is a topological group. For every $j \in I$, the projection map $pr_j : G \to G_j$ is an open morphism.

Proof We show that $\varkappa(g,h) = g^{-1}h$ is continuous on $G \times G$. For each j the map $\varkappa_j : G_j \times G_j \to G_j$
$$(g_j, h_j) \mapsto g_j^{-1} h_j$$
is continuous, whence

$$\begin{tikzcd} G \times G \arrow[r, "\varkappa"] \arrow[d, "pr_j \times pr_j"', "cts"] \arrow[dr, "cts"] & G \arrow[d, "pr_j"] \\ G_j \times G_j \arrow[r, "\varkappa_j", "cts"'] & G_j \end{tikzcd}$$

$pr_j \circ \varkappa$ is continuous for each j

Hence $\varkappa$ is continuous by the universal property of the product topology. Hence G is a topological group.

The maps pr_j are group homomorphisms.

They are also continuous and open, hence they are open morphisms. $\square$

Example Put $H = \{0,1\}$ with the discrete topology, and with $0+1=1$ $=1+0$, $0+0=0$, $1+1=0$.

Then H is an abelian topological grp. Hence the Cantor set

$$C := H^{\mathbb{N}} = \prod_{k\in\mathbb{N}} H$$

is a compact abelian grp.

Elements are sequences $(c_j)_{j\in\mathbb{N}}$ with $c_j = 0,1$. #

5. Prop. Let G be a topological grp, let $H \leq G$ be a subgrp. Then H is a topological grp with respect to the subspace topology. Moreover, the closure $\overline{H} \leq G$ is also a subgrp. If $H \trianglelefteq G$ is normal, so is $\overline{H}$.

proof The first claim is clear (is it?)

Consider $\varkappa(g,h) = g^{-1}h$. Since $\varkappa$ is continuous,

$$\varkappa(\overline{H\times H}) \overset{(!)}{=} \varkappa(\overline{H}\times\overline{H}) \subseteq \overline{\varkappa(H)} = \overline{H}$$

i.e. $g,h\in\overline{H} \Rightarrow g^{-1}h\in\overline{H}$. Hence $\overline{H}$ is a subgrp.

If $a\in G$, then put $\gamma_a(g) = a g a^{-1}$. Then

$\gamma_a(\overline{H}) \subseteq \overline{\gamma_a(H)}$, hence $a\overline{H}a^{-1} \subseteq \overline{H}$ if H is normal.

6. **Prop** Let G be a Hausdorff topological group, let $A \subseteq G$ be a subset. Then the <u>centralizer of A</u>

$$\mathrm{Cen}_G(A) = \{g \in G \mid ga = ag \text{ for all } a \in A\}$$

is a closed subgroup. In particular, the <u>center</u> $\mathrm{Cen}_G(G) = \mathrm{Cen}(G)$ is closed in G.

<u>proof</u> Put $f_a(g) = [g,a] = gag^{-1}a^{-1}$. Then $f_a^{-1}(e) = \{g \in G \mid ga = ag\}$ is closed, since $\{e\} \subseteq G$ is closed, and

$$\mathrm{Cen}_G(A) = \bigcap_{a \in A} f_a^{-1}(e)$$

is also closed. $\square$

7. **Prop** If G is a Hausdorff topological group and if $A \subseteq G$ is an abelian subgroup, then $\bar{A}$ is also abelian.

<u>proof</u> The map $[\,,\,] : G \to G$,

$$(g,h) \mapsto [g,h]$$

is constant on A and hence constant on $\bar{A}$. $\square$

8. <u>Lemma</u> Let G be a topological grp and let $X, U \subseteq G$ be subsets. If $U \subseteq G$ is open, then $XU = \{xu \mid x \in X, u \in U\}$ is also open. Similarly UX is open.

<u>proof</u> $XU = \bigcup_{x \in X} xU$ and each set $xU = \lambda_x(U)$ is open. $\square$

9. <u>Prop</u> Let G be a topological group and let H be a subgrp.

(i) H is open in G iff H contains a nonempty open subset $U \subseteq G$

(ii) If H is open, then H is also closed

(iii) H is closed iff there is an open subset $U \subseteq G$ with $H \cap U = \phi$, such that $H \cap U$ is closed in U.

<u>proof</u> (i) Suppose $U \neq \phi$ is open, $U \subseteq H$. Then $H = UH$ is open by Lemma §1.8. $\square$

(ii) Suppose $H \subseteq G$ is open. Then $G - H = \bigcup \{gH \mid g \in G - U\}$ is open by Lemma §1.8. $\square$

(iii) Suppose $U \cap H \neq \phi$ is closed in U.

Then $U \cap H$ is closed in $\underbrace{U \cap \overline{H}}_{\text{closed in } U}$.

Hence we may replace G by $\overline{H} \subseteq G$.

Then $\underbrace{U - (U \cap H)}_{= U - H}$ is open in G. But

$H \subseteq G$ is dense, hence $U - H = \phi$, whence $U \subseteq H$.

Thus H is open and closed, whence $H = G = \overline{H}$ $\square$

<u>Corollary A</u> Let G be a topological group, let $U \subseteq G$ be open and nonempty, let $H = \langle U \rangle$ denote the subgroup generated by U. Then H is open (and closed) in G.

<u>proof</u> This follows from §1.9 (i). $\square$

<u>Corollary B</u> Let G be a Hausdorff top. group, let $H \subseteq G$ be a subgrp. If H is locally compact (as a subspace) then H is closed in G. In particular, every discrete subgrp. $H \subseteq G$ is closed.

<u>proof</u>. Let $C \subseteq H$ be a compact identity neighborhood (= neighborhood of e). Then there is $U \subseteq G$ open with $e \in U$ and $H \cap U \subseteq C$. Thus $U \cap H = U \cap C$ is closed in U. Now use §1.9. (iii). $\square$

<u>Note:</u> if $A, B \subseteq G$ are closed subsets in a topological group G, then $AB = \{ab \mid a \in A, b \in B\}$ need not be closed. E.g. $G = \mathbb{R}$, $A = \mathbb{Z}$, $B = \sqrt{2} \cdot \mathbb{Z}$. Then $A + B = \{k + l\sqrt{2} \mid k, l \in \mathbb{Z}\}$ is <u>dense</u> in $\mathbb{R}$! ($\to$ Exercise)

We recall a useful fact from topology.

10. <u>Lemma</u> (Wallace's Lemma)

Let $X_1, \dots, X_m$ be Hausdorff spaces, let $A_j \subseteq X_j$ be compact, $j = 1, \dots, m$. Let $W \subseteq X_1 \times \dots \times X_m$ be open, with $A_1 \times \dots \times A_m \subseteq W \subseteq X_1 \times \dots \times X_m$. Then there are open sets $U_j \subseteq X_j$, $j = 1, \dots, m$ such that $A_1 \times \dots \times A_m \subseteq U_1 \times \dots \times U_m \subseteq W$.

A proof of Wallace's Lemma is given in R. Engelking, General Topology, § 3.2.10 or in my course GATG, SoSe 24, Satz 2.15 (available on www page).

11. <u>Prop</u> Let G be a Hausdorff topological group, let $A, B \subseteq G$ be closed subsets. If A or B is compact, then $AB \subseteq G$ is closed.

<u>proof</u> Let $g \in G - AB$, then $A^{-1}g \cap B = \emptyset$.

Consider $\varkappa : G \times G \to G$, $(x, g) \longmapsto x^{-1}g$. Thus

$\varkappa(A \times \{g\}) \subseteq \underbrace{G - B}_{\text{open}}$. If A is compact, then Wallace's Lemma shows that there is an open set $V \subseteq G$ with $g \in V$ and $\varkappa(A \times V) \subseteq G - B$, i.e. $A^{-1}V \cap B = \emptyset \rightsquigarrow V \cap AB = \emptyset$. $\square$

Now we consider quotients. If G is a topological group and if $H \subseteq G$ is a subgroup, we have the natural map $G \xrightarrow{q} G/H$, $g \longmapsto gH$. We put the <u>quotient topology</u> on G/H. Hence $W \subseteq G/H$ is open iff $q^{-1}(W) \subseteq G$ is open.

12. __Prop.__ Let G be a topological group, let $H \subseteq G$ be a subgroup. Then $q: G \to G/_H$ is a continuous and open map. The space $G/_H$ is Hausdorff iff $H \subseteq G$ is closed.

__proof.__ By definition q is continuous. Let $U \subseteq G$ be open. Then $q^{-1}(q(U)) = UH$ is open by §1,8, hence $q(U)$ is open. If $G/_H$ is Hausdorff, then $\{eH\} = \{H\} \subseteq G/_H$ is closed, hence $q^{-1}(\{H\}) = H$ is closed as well.

If H is closed, then $W = \{(x,y) \in G \times G \mid x^{-1}y \in G - H\}$ is open and $q \times q\,(W) = \{(xH, yH) \mid xH \neq yH\}$ is also open (because $q \times q$ is open!). Hence $G/_H$ is Hausdorff. $\square$

13. __Corollary__ Let G be a topological group, let $N \trianglelefteq G$ be a normal subgroup. Then $G/_N$ is a topological group (for the quotient topology of $q: G \to G/_N$), which is Hausdorff iff $N \subseteq G$ is closed.

The map $q: G \to G/N$ is an open morphism. In particular, $G/\overline{N}$ is a Hausdorff topological group (and so is $G/\overline{\{e\}}$).

proof Consider the commutative diagram

$$\begin{tikzcd} G \times G \arrow[r, "\chi"] \arrow[d, "q \times q"'] \arrow[dr, "\text{cts}"] & G \arrow[d, "q"] \\ G/N \times G/N \arrow[r, "\tilde{\chi}"] & G/N \end{tikzcd}$$

$\chi(g,h) = g^{-1}h$

$\tilde{\chi}(gN, hN) = g^{-1}hN$

Then $q \times q$ is open (since q is open), hence a quotient map. Thus $\tilde{\chi}$ is continuous. $\square$

Recall: a topological space X is connected if ϕ, X are the only clopen (= closed and open) subsets of X. A subset $Y \subseteq X$ is connected if Y is connected in the subspace topology. If $Y \subseteq X$ is connected, then $\overline{Y}$ is also connected. The continuous image of a connected set is connected. Products of connected sets are connected.

A topological space X is <u>totally disconnected</u> if all nonempty connected subsets of X is a singleton $\{x\} \subseteq X$.

Recall also: For any topological space X and every $x \in X$ there is a largest connected subset $C(x) \subseteq X$ containing x, namely

$$C(x) = \bigcup \{ Y \subseteq X \mid x \in Y \text{ and } Y \text{ is connected} \}$$

the <u>connected component</u> of x.

14. <u>Def</u> Let G be a topological grp. The <u>identity component</u> is

$$G^0 = C(e) = \bigcup \{ Y \subseteq G \mid e \in Y, \; Y \text{ is connected} \}$$

Note: for $g \in G$ we have then $C(g) = gG^0$

15. <u>Prop</u> Let G be a topological grp. Then G^0 is a closed normal subgrp and G/G^0 is a totally disconnected Hausdorff topological grp.

<u>proof</u> G^0 is closed and connected.

For $\varkappa(g,h) = g^{-1}h$, the image
$\varkappa(G^0 \times G^0) \subseteq G$ is connected, whence $\varkappa(G^0 \times G^0) \subseteq G^0$.
Hence G^0 is a subgrp. For every $a \in G$
we have $\gamma_a(G^0) \subseteq G^0$ since $\gamma_a(g) = aga^{-1}$
is continuous. Hence $G^0 \trianglelefteq G$, and

G/G^0 is Hausdorff (because G^0 is closed, §1.13.

Consider $p : G \to G/G^0$, $g \mapsto p(g) = gG^0$,
put $H = \left(G/G^0\right)^0$. <u>Claim:</u> $H = \{G^0\}$.
Put $N = p^{-1}(H) \rightsquigarrow N \subseteq G$ is closed and normal,
with $G^0 \subseteq N$. Consider

$$\begin{tikzcd}
N \arrow[r, "p|_N^H"] \arrow[d, hook] & H \arrow[d, hook] \\
G \arrow[r, "p", "(\text{open})"'] & G/G^0
\end{tikzcd}$$

Since $N = p^{-1}(H)$, the map $p|_N^H$ is also
open ($p|_N^H(U \cap N) = p(U) \cap H$ is open in
H if $U \subseteq G$ is open) and hence a
cp. Dugundji $\underline{\text{VI}}$ 2.1 (2)

quotient map. Suppose $V \subseteq N$ is clopen in N, with $e \in V$. Then $vG_0 \subseteq V$ for all $v \in V$, since vG_0 is connected. Thus $V = p^{-1}(p(V))$ and $p(V)$ is clopen in H.(*) Hence $p(V) = H \Rightarrow V = N$. Thus N is connected, $N \subseteq G^0 \Rightarrow N = G^0$ and $H = \{G^0\}$. But then every $\overbrace{\text{connected}}^{\text{nonempty}}$ compont in G consists of one element. $\square$

Cor. Let $f: G \to K$ be a morphism of topological grps. If K is totally disconnected, then $G^0 \subseteq \ker(f)$, and f factors as

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "p"'] & K \\ G/G^0 \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

and $\bar{f}$ is a morphism.

proof $f(G^0) \subseteq K$ is connected, where $f(G^0) = \{e_K\}$

$\square$

(*) Here we use that $q|_N^H$ is a quotient map!

Remark In the proof above we used that G is totally disconnected iff $G^0 = \{e\}$.

Example $(\mathbb{Q}, +)$ and $(\mathbb{Z}/2)^{\mathbb{N}}$ are totally disconnected topological grps.

16. *Def* A pseudo metric d on a set X is a map $d : X \times X \to \mathbb{R}$ such that

$$\left.\begin{array}{l} \circ\ d(x,y) = d(y,x) \geq 0 \\ \circ\ d(x,x) = 0 \\ \circ\ d(x,z) \leq d(x,y) + d(y,z) \end{array}\right\} \text{for all } x,y,z \in X$$

If $d(x,y) = 0 \Rightarrow x = y$, then d is called a metric.

A pseudo metric d on a group G is called left-invariant if $d(g,h) = d(ag, ah)$ holds for all $a, g, h \in G$. (Right-invariant is defined analogously, $d(g,h) = d(ga, ha)$).

A length function l on a group G is a map $l : G \to \mathbb{R}$ such that

- $l(e) = 0$
- $l(g) = l(g^{-1}) \geq 0$
- $l(gh) \leq l(g) + l(h)$.

If l is a length function, then

$d(g,h) = l(g^{-1}h)$ is a left-invariant

pseudo metric. This is a metric iff $l(a) \Rightarrow a = e$.

pf $d(g,h) = l(g^{-1}h) = l(h^{-1}g) = d(h,g) \geq 0$

$d(g,g) = l(e) = 0$

$d(g,h) = l(g^{-1}h) \leq l(g^{-1}a) + l(a^{-1}h) = d(g,a) + d(a,h)$

$d(ag,ah) = l(g^{-1}a^{-1}ah) = l(g^{-1}h) = d(g,h)$

Note also: $H = \{h \in G \mid l(h) = 0\} \subseteq G$ is a subgroup of G.

17 <u>Theorem</u> (Birkhoff - Kakutani)

(Garrett Birkhoff, 1911-1996 American Math.
Shizuo Kakutani, 1911-2004 Japanese Math.)

Let G be a Hausdorff topological group.

The following are equivalent.

(i) the topology of G is metrizable.

(ii) the topology of G is metrizable by a left invariant metric

(iii) $e \in G$ has a countable neighborhood basis $\mathcal{N}$ (i.e. a countable collection $\mathcal{N}$ of neighborhoods of e such that for every neighborhood W of e there is some $V \in \mathcal{N}$ with $V \subseteq W$.) #

Note that (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii). For the rest we need a lemma.

We call an identity neighborhood V <u>symmetric</u> if $V = V^{-1}$ $(= \{v^{-1} \mid v \in V\})$. If W is any identity neighborhood, then $V = W \cap W^{-1} \subseteq W$ is a symmetric identity neighborhood.

18. <u>Lemma</u> Let G be a topological group, let $(K_m)_{m \in \mathbb{Z}}$ be a family of (symmetric) identity neighborhoods such that $K_m \cdot K_m \cdot K_m \subseteq K_{m+1}$ holds for all $m \in \mathbb{Z}$, with $\langle \bigcup_{m \in \mathbb{Z}} K_m \rangle = G$. Then there is a continuous length function $l : G \to \mathbb{R}$ such that

$$\{g \in G \mid l(g) < 2^m\} \subseteq K_m \subseteq \{g \in G \mid l(g) \leq 2^m\}$$

holds for all $m \in \mathbb{Z}$. In particular,

$$\bigcap_{m \in \mathbb{Z}} K_m = \{g \in G \mid \ell(g) = 0\}.$$

<u>proof</u> For every $g \in G$ there exist $m_1, \ldots, m_k \in \mathbb{Z}$ with $g \in K_{m_1} \cdots K_{m_k}$. We put

$$\ell(g) = \inf\{t \geq 0 \mid t = 2^{m_1} + \ldots + 2^{m_k} \text{ and } g \in K_{m_1} \cdots K_{m_k}\}$$
$k \geq 1$

Since each K_m is symmetric, $\ell(g) = \ell(g^{-1}) \geq 0$. If $g \in K_{m_1} \cdots K_{m_r}$, $h \in K_{n_1} \cdots K_{n_s}$, then $gh \in K_{m_1} \cdots K_{n_s}$, whence $\ell(gh) \leq \ell(g) + \ell(h)$. Since $e \in K_m$ for all m, $\ell(e) = 0$. Hence ℓ is a length function. Also, $\ell(g) \leq 2^m$ holds for all $g \in K_m$.

<u>Claim: ℓ is continuous.</u> Let $g \in G$, $\varepsilon > 0$. Choose $m \in \mathbb{Z}$ with $2^m \leq \varepsilon$. Since $\ell(a) \leq 2^m$ for all $a \in K_m$, we have $|\ell(g) - \ell(h)| \leq 2^m \leq \varepsilon$ for all $h \in gK_m$. Hence ℓ is continuous at g.

Suppose finally that $\ell(g) < 2^m$. We claim that $g \in K_m$. Then are $m_1, \dots, m_k$ such that $g \in K_{m_1} \cdots K_{m_k}$ and $2^{m_1} + \dots + 2^{m_k} < 2^m$.

<u>Claim</u> If $2^{m_1} + \dots + 2^{m_k} < 2^m$, then $K_{m_1} \cdots K_{m_k} \subseteq K_m$.

<u>proof of this claim.</u> Note first that $m_j < m$ for $j = 1, \dots, k$. Thus $K_{m_j} \subseteq K_{m-1}$, hence the claim is true for $k = 1, 2, 3$. For $k \geq 4$ we proceed by induction on k.

<u>Case (a)</u> $2^{m_1} + \dots + 2^{m_k} \leq 2^{m-1}$. Then

$2^{m_1} + \dots + 2^{m_{k-1}} < 2^{m-1}$, hence $K_{m_1} \cdots K_{m_{k-1}} \subseteq K_{m-1}$

and $K_{m_1} \cdots K_{m_{k-1}} K_{m_k} \subseteq K_{m-1} K_{m-1} \subseteq K_m$ (v)

<u>Case (b)</u> $2^{m-1} < 2^{m_1} + \dots + 2^{m_k} < 2^m$, $k \geq 4$.

Then $m_1 < m-1$. Pick the largest j with $2^{m_1} + \dots + 2^{m_j} < 2^{m-1}$

$$\left.\begin{array}{l} \Rightarrow K_{m_1} \cdots K_{m_{j-1}} \subseteq K_{m-1} \\ K_{m_j} \subseteq K_{m-1} \\ K_{m_{j+1}} \cdots K_{m_k} \subseteq K_{m-1} \end{array}\right\} K_{m_1} \cdots K_{m_k} \subseteq K_m$$

(also ok for $j = 1$ or $j = k$, check.)

Hence the claim is true, whence $\{g \in G \mid \ell(g) < 2^m\} \subseteq K_m$.

In particular, $\{g \in G \mid \ell(g) = 0\} \subseteq K_m$ holds for all $m \in \mathbb{Z}$, whence $\{g \in G \mid \ell(g) = 0\} \subseteq \bigcap_{m \in \mathbb{Z}} K_m$.

If $g \in \bigcap_{m \in \mathbb{Z}} K_m$, then $\ell(g) \leq 2^m$ for all m, hence $\ell(g) = 0$. $\square$

Proof of the Birkhoff-Kakutani-Theorem §2.17.

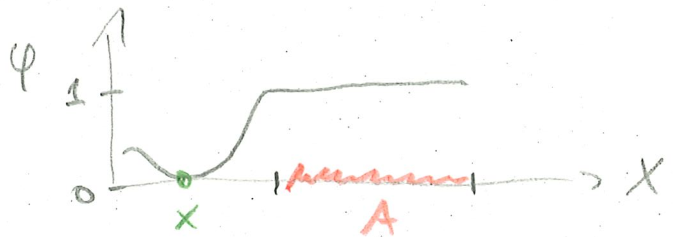
It remains to show that (iii) $\Rightarrow$ (ii). Let $\mathcal{N} = \{V_m \mid m \in \mathbb{N}\}$ be a countable neighborhood basis.

Put $G = K_m$ for all $m \geq 1$. For $m \leq 0$ we choose inductively symmetric identity neighborhoods K_m with $K_m \subseteq V_{-m}$ and $K_m \cdot K_m \cdot K_m \subseteq K_{m+1}$.

This is possible since $G \times G \times G \to G$, $(x, y, z) \mapsto xyz$ is continuous. Let ℓ be a continuous length function as in Lemma §1.18, put $d(g, h) = \ell(g^{-1}h)$.

If $d(g, h) < 2^m$, then $h \in g K_m$. Hence every open subset of G is also open with respect to the metric d. Since d is continuous, every d-open set is also open in G. Thus d metrizes the topology on G, and d is left invariant. $\square$

The next result is possibly more important than Thm §1.17 itself. Recall that a Hausdorff space X is called <u>completely regular</u> or a $T_{3\frac{1}{2}}$-space or a <u>Tychonoff space</u> if the following holds: for every closed $A \subseteq X$ and every $x \in X - A$ there is a continuous $\varphi : X \to [0,1]$ with $\varphi(x) = 0$ and $\varphi(a) = 1$ for all $a \in A$.



19. <u>Prop.</u> Every Hausdorff topological group G is completely regular.

<u>proof</u> Let $A \subseteq G$ be closed and $g \in G - A$. Since G is homogeneous, we may assume $g = e$. Put $W = G - A$.

and $K_m = G$ for all $m \geq 1$, let K_0 be a symmetric identity neighborhood with $K_0 \subseteq V$, then choose K_m inductively with $K_m \cdot K_m \cdot K_m \subseteq K_{m+1}$ for all $m < 0$. Let $l : G \to \mathbb{R}$ be a continuous length function as in Lemma §2.17. Then $l(e) = 0$ and $\{ g \in G \mid l(g) < 2^0 \} \subseteq K_0 \subseteq W$. If we put $\varphi(g) = \min \{ l(g), 1 \}$ then φ is continuous, and $\varphi(a) = 1$ for all $a \in A$. $\square$

20. <u>Def</u> Let G be a Hausdorff topological group. A sequence $(g_k)_{k \in \mathbb{N}}$ is called a <u>left Cauchy sequence</u> if for every identity neighborhood $V \subseteq G$ there is some $n \in \mathbb{N}$ such that $\underbrace{g_j^{-1} g_k \in V}_{\Leftrightarrow g_k \in g_j V}$ holds for all $j, k \geq n$.

Note: every convergent sequence is a left Cauchy sequence: if $\lim_k g_k = g$

choose an identity nbhd W such that $W^{-1}W \subseteq V$.

For all $k \geq n$, we have $g_k \in gW$. Then

$g_j^{-1} g_k \in W^{-1}g^{-1}gW \subseteq V$ for all $j, k \geq n$. $\square$

Note also: if G is metrizable by a left invariant metric d, then a d-Cauchy sequence is the same as a left Cauchy sequence

21. __Prop__ Let G be a metrizable topological group.

The following are equivalent.

(i) some left invariant metric d that metrizes G is complete

(ii) every left invariant metric d that metrizes G is complete

(iii) every left Cauchy sequence converges. $\square$

A $\underbrace{\text{group}}_{\text{metrizable topological}}$ satisfying these equivalent conditions is called __Weil complete.__ (A. Weil, 1906-1998, French math.)

#

22. <u>Def</u> Let X be a topological space. A subset $N \subseteq X$ is <u>nowhere dense</u> if $\overline{N}$ has empty interior (= contains no open set $U \neq \emptyset$). Equivalently, $X - \overline{N}$ is dense in X. Subsets of nowhere dense sets are again nowhere dense.

A subset $M \subseteq X$ is <u>meager</u> if M is the union of countably many nowhere dense sets. Equivalently, there are open dense sets $U_k \subseteq X$, $k \in \mathbb{N}$, with

$$M \cap \bigcap_{k \in \mathbb{N}} U_k = \emptyset.$$

Countable unions of meager sets are meager.

A topological space X is called a <u>Baire space</u> if every meager set $M \subseteq X$ has empty interior; equivalently: if $U_k \subseteq X$ are open and dense, $k \in \mathbb{N}$, then $\bigcap_{k=0}^{\infty} U_k$ is dense.

If $X \neq \emptyset$ is a Baire space, then $X \subseteq X$ is not meager.

[R. Baire, French math., 1874–1922]

! Warning! People working in descriptive set theory call $\mathbb{N}^{\mathbb{N}}$ <u>The Baire space</u>. This is <u>confusing</u>, but note that $\mathbb{N}^{\mathbb{N}}$ is a Baire space in our sense.

23. **Prop** (Baire) Let $X \neq \emptyset$ be a Baire space, let $A_k \subseteq X$ be closed for all $k \in \mathbb{N}$. If $\bigcup_{k=0}^{\infty} A_k = X$, then some A_k has nonempty interior.

proof Otherwise, X would be meager. □

24. **Theorem** (Baire's Category theorem)
Every locally compact space and every completely metrizable space is a Baire space.

proof Let $(U_k)_{k=0}^{\infty}$ be open dense sets in X, let $W \subseteq X$ be a nonempty open set. We claim that $W \cap \bigcap_{k=0}^{\infty} U_k \neq \emptyset$.

Consider $W \cap U_0 \neq \emptyset$. If X is locally compact, there is an open set $Y_0 \neq \emptyset$, with $\overline{Y_0} \subseteq W \cap U_0$. Then pick $Y_1 \neq \emptyset$ with $\overline{Y_1} \subseteq Y_0 \cap U_1$ etc. Thus $\overline{Y_k} \subseteq U_k \cap W$ and $\bigcap_{k=0}^{\infty} \overline{Y_k} \neq \emptyset$ by compactness, hence $W \cap \bigcap_{k=0}^{\infty} U_k \neq \emptyset$.

If X is completely metrizable, there is $\varepsilon_0 < 1$ with $\overline{B}_{\varepsilon}(x_0) \subseteq W \cap U_0$, $\varepsilon_1 < \frac{1}{2}$ with $B_{\varepsilon_1}(x_1) \subseteq B_{\varepsilon}(x_0) \cap U_1$ etc $\leadsto (x_k)_{k=0}^{\infty}$ is a Cauchy sequence $\leadsto \bigcap_{k=0}^{\infty} B_{\varepsilon_k}(x_k) \neq \emptyset$ etc. □

Recall that a Hausdorff space X is <u>σ-compact</u> if there are compact subsets $A_k \subseteq X$, $k \in \mathbb{N}$, with $X = \bigcup_{k=0}^{\infty} A_k$.

- Every separable locally compact space and every compact space is σ-compact.
- $\mathbb{Q} = X$ is σ-compact, but not locally compact.

25. <u>Theorem</u> (Open mapping theorem)

Let $f: G \to K$ be a morphism of Hausdorff topological groups. If G is σ-compact and if $f(G) \subseteq K$ is not meager, then f is open and K is locally compact.

<u>proof</u> Assume in addition that f is <u>injective</u>.

Put $G = \bigcup_{k=0}^{\infty} A_k$, with A_k compact. Then $A_k \xrightarrow{f_k} f(A_k)$ is a homeomorphism for every k (because A_k is compact and f_k is injective). Also, $f(A_k) \subseteq K$ is closed. Hence there is some m such that $f(A_m)$ has nonempty interior (since $f(G)$ is not meager). Hence

$\emptyset \neq W \subseteq f(A_k)$, W open. Then $f(G) \subseteq K$

(Thus K is locally compact.)

is open (by §1.9). Put $U = f^{-1}(W)$. Since

$f_m : A_m \to f(A_m)$ is a homeomorphism,

$U \to W$ is also a homeomorphism.

Thus $f(G) \to G$ is a group homomorphism

$f(g) \mapsto g$

which is continuous at some point, hence continuous by Exercise 1.1. Hence $G \xrightarrow{f} f(G)$ is a homeomorphism. Since $f(G) \subseteq K$ is open, $G \xrightarrow{f} K$ is an open morphism.

In general, put $N = \ker(f)$ and consider

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "q"'] & K \\ G/N \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

$q(g) = gN$.

Then $\bar{f}$ is open by the previous argument, hence $f = \bar{f} \circ q$ is an open morphism.

(Here we use that G/N is also σ-compact,

$G/N = \bigcup_k \underbrace{q(A_k)}_{\text{compact}}$.)

$\square$

26. <u>Cor</u> (The Closed Graph Theorem)

Let G, K be Hausdorff topological groups, let $f: G \to K$ be a group homomorphism. Assume that G, K are σ-compact and that G is a Baire space. The following are equivalent:

(i) f is a morphism (i.e. continuous)

(ii) The graph of f is closed in $G \times K$.

<u>proof</u> (i) $\Rightarrow$ (ii): holds since K is a Hausdorff space; graphs of continuous maps into Hausdorff spaces are closed.

(ii) $\Rightarrow$ (i): Put $M = \{(x, f(x)) \mid x \in G\} \subseteq G \times K$. Then $M \subseteq G \times K$ is a closed subgroup, hence σ-compact (since G, K are σ-compact). Consider $h: M \to G$, $(x, f(x)) \mapsto x$. This is a bijective morphism, hence open by §2.25, hence a homeomorphism. Thus $x \mapsto (x, f(x))$ is also continuous. $\square$

Unions of countably many meager sets are again meager. This need not be true for arbitrary unions (e.g. $\mathbb{R} = \bigcup_{r\in\mathbb{R}} \underbrace{\{r\}}_{\text{meager}}$). However, we have the following.

27. Theorem (Banach's Category Theorem)

[S. Banach, Pol. math., 1892–1945]

Let X be a topological space, let $\mathcal{U}$ be a set of open meager subsets of X. Then $\bigcup\mathcal{U} \subseteq X$ is also meager.

Proof [Wlog $\mathcal{U}\neq\emptyset$.] We consider the collection $\mathcal{G}$ consisting of all sets $\mathcal{W}$ of the following form: the elements of $\mathcal{W}$ are open subsets $W \subseteq X$, each $W \in \mathcal{W}$ is contained in some $U \in \mathcal{U}$, and if $W_1, W_2 \in \mathcal{W}$ are distinct, then $W_1 \cap W_2 = \emptyset$.

We order $\mathcal{G}$ partially by inclusion "$\subseteq$".

If $\mathcal{B} \subseteq \mathcal{G}$ is a linearly ordered subset, then $\bigcup\mathcal{B} \in \mathcal{G}$ is an upper bound for $\mathcal{B}$. By Zorn's Lemma, $\mathcal{G}$ has maximal elements.

Let $\mathcal{W} \in \mathcal{G}$ be "$\subseteq$"-maximal. Put $M = \overline{\bigcup \mathcal{U}} - \bigcup \mathcal{W}$. Thus $M \subseteq X$ is closed. We claim that M has empty interior. If $V \subseteq M$ is open, $V \neq \phi$, then $V \cap U \neq \phi$ for some $U \in \mathcal{U}$. But then $\mathcal{W} \cup \{V \cap U\} \in \mathcal{G}$ ↯. Hence M has empty interior and M nowhere dense.

Each $W \in \mathcal{W}$ is meager, whence $W = \bigcup_{k=0}^{\infty} N_{W,k}$ each $N_{W,k}$ nowhere dense. Put $N_k = \bigcup_{W \in \mathcal{W}} N_{W,k}$

We claim N_k is also nowhere dense. If $\phi \neq V$ is open, $V \subseteq \overline{N_k}$, then there is $W \in \mathcal{W}$ with $V \cap W \neq \phi$. Now $W \cap \overline{N_k} \subseteq \overline{N_{W,k}}$, whence $V \cap W \subseteq \overline{N_{W,k}}$ ↯. Hence N_k is nowhere dense, and $\bigcup \mathcal{W} = \bigcup_{k=0}^{\infty} N_k$ is meager. Finally,

$$\bigcup \mathcal{U} \subseteq \underbrace{(\bigcup \mathcal{W})}_{\text{meager}} \cup \underbrace{M}_{\text{meagr}}$$ is also meager. $\square$

We apply this to topological groups.

28. Let G be a topological grp. The following are equivalent: (i) G is a Baire space.

(ii) G is not meager (in itself)

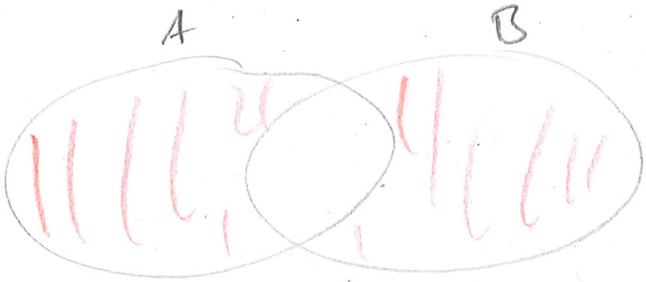
(iii) G contains a non-meager subset.

proof (i) $\Rightarrow$ (ii) since $G \neq \emptyset$.

(ii) $\Leftrightarrow$ (iii). clear.

$\neg$(i) $\Rightarrow$ $\neg$(ii): If G is not a Baire space, there is a meager set $M \subseteq G$ with nonempty interior U. Hence U is meager. But then $G = \bigcup_{g \in G} gU$ is also meager by §1.27. $\square$

Recall that the symmetric difference of two sets A, B is $A \triangle B = (A - B) \cup (B - A)$
$= (A \cup B) - (A \cap B)$



$A \triangle B$

29. Def A subset $A \subseteq X$ in a topological space is called Baire measurable if there is an open set $U \subseteq X$ such that $M = U \triangle A$ is meager.

(Baire measurable sets are also called <u>almost open</u> or subsets having the <u>Baire property</u>.)

Note: there is no measure here...

30. <u>Theorem</u> (<u>Pettis' Lemma</u>) Let G be a topological grp, let $A \subseteq G$ be a Baire measurable subset. If A is not meager, then $A^{-1}A \subseteq G$ is an identity neighborhood.

<u>proof</u> Let $U \subseteq G$ be open with $M = A \Delta U$ meager. Then U is not meager, since $A \subseteq U \cup M$ is not meager. Hence $U \neq \phi$. Choose $g \in U$ and an identity nbhd $V \subseteq G$ with $gV^{-1} \subseteq U$

<u>Claim</u>: <u>$V \subseteq A^{-1}A$.</u>

<u>pf</u> Let $h \in V$. Then $gh^{-1} \in U$, whence $g \in U \cap Uh$.

Moreover, $(A \cap Ah) \Delta (U \cap Uh) \overset{(*)}{\subseteq} (A \Delta U) \cup (Ah \Delta Uh)$
$$= \underbrace{(A \Delta U)}_{M} \cup \underbrace{(A \Delta U)}_{M} h$$

is meager.

$\ulcorner_{(*)}$ Suppose $a \in (A \cap Ah) - (U \cap Uh)$
If $a \in U$, then $a \notin Uh$, hence $a \in Ah \Delta Uh$
If $a \notin U$, then $a \in A \Delta U$
In any case, $a \in (A \Delta U) \cup (Ah \Delta Uh)$ $\lrcorner$

If $A \cap Ah = \phi$, then $U \cap Uh$ is meager ↯
(G is a Baire space by § 1.28). Hence
there is $a \in A \cap Ah$ so $a = bh$ for some $b \in A$
$\Rightarrow h = b^{-1}a \in AA^{-1}$ □

Cor Let G be a topological group, let
$H \subseteq G$ be a subgrp. If H contains a non-meager Baire measurable set, then $H \subseteq G$ is
open and closed. □

31. Def A map $f: X \to Y$ between topological spaces
is Baire measurable if $f^{-1}(U) \subseteq X$ is Baire
measurable for every open set $U \subseteq Y$.
A topological space is Lindelöf if every open
covering has a countable subcovering.
Every compact space, every σ-compact space and
every space with a countable base (= 2nd countable
space) is Lindelöf. Every Polish space
(= completely metrizable separable space) is Lindelöf.

32. <u>Theorem</u> Let G, K be topological groups, let $f: G \to K$ be a group homomorphism. Assume that G is not meager and that K is Lindelöf. The following are equivalent.

(i) f is a morphism (i.e. continuous).

(ii) f is Baire measurable.

<u>proof</u> Every continuous map is Baire measurable, hence (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (i): $\overline{f(G)} \subseteq K$ is also Lindelöf, hence wlog $\overline{f(G)} = K$. Let $V \subseteq K$ be an identity neighborhood. Then is an identity neighborhood $W \subseteq K$ with $W^{-1}W \subseteq V$. Put $E = f^{-1}(W)$. Then E is Baire measurable. Since $K = \bigcup_{g \in G} f(g)W$

(because $a \in K \Rightarrow (aW^{-1}) \cap (f(G)) \neq \emptyset \Rightarrow a \in f(G)W$)

and since K is Lindelöf, $K = \bigcup_{k=0}^{\infty} f(g_k)W$ for certain $g_k \in G$. Hence $G = \bigcup_{k=0}^{\infty} g_k E$

Since G is not meager, E is not meager. Hence $E^{-1}E = U$ is an identity nbhd in G, with $f(U) \subseteq V$.

Thus f is continuous at e_G, and hence continuous by Exercise 1.1.

$\square$

33. <u>Def</u> A set $\mathcal{A}$ of subsets of a set X is called a <u>σ-algebra</u> if the following hold.
(i) $\phi \in \mathcal{A}$. (ii) $A \in \mathcal{A} \Rightarrow X - A \in \mathcal{A}$.
(iii) $A_k \in \mathcal{A}$ for all $k \in \mathbb{N} \Rightarrow \bigcup_{k=0}^{\infty} A_k \in \mathcal{A}$.

If X is a topological space, then there is a unique smallest σ-algebra $\mathcal{B}$ containing all open subsets of X. These are the <u>Borel sets</u>.

<u>Lemma</u> Let $\mathcal{C} = \{A \subseteq X \mid A \text{ is Baire measurable}\}$. Then $\mathcal{C}$ is a σ-algebra (which contains all Borel sets).

<u>proof</u> $\phi \in \mathcal{C}$ (v). Every open set $U \subseteq X$ is in $\mathcal{C}$, since $U \Delta U = \phi$ is meager.

Suppose $A \in \mathcal{C} \Rightarrow$ there is an open set $U \subseteq X$ such that $M = A \Delta U$ is meager. Put $V = X - \overline{U}$, $B = X - A$. Then

$$V \Delta B = \overline{U} \Delta A \subseteq \underbrace{(U \Delta A)}_{\text{meager}} \cup \underbrace{(\overline{U} - U)}_{\text{nowhere dense}}$$

hence $B \in \mathcal{C}$.

Suppose $A_k \in \mathcal{C}$ for all $k \in \mathbb{N}$. Put $M_k = A_k \Delta U_k$ U_k open, M_k meager.

Put $U = \bigcup_{k=0}^{\infty} U_k$, $M = \bigcup_{k=0}^{\infty} M_k$ $A = \bigcup_{k=0}^{\infty} A_k$

open meager

Then $A_k - U \subseteq A_k - U_k \subseteq M_k \subseteq M$, whence $A - U \subseteq M$. Similarly $U - A \subseteq M$, whence $U \Delta A \subseteq M$ is meager $\Rightarrow A \in \mathcal{C}$. $\square$

Recall that a G_δ-set is an intersection of countably many open sets.

Cor Let G be a topological subgrp, let $H \subseteq G$ be a dense subgroup. If G is not meager and if H is a G_δ-set in G, then $H = G$.

proof Put $H = \bigcap_{k=0}^{\infty} U_k$, U_k open. Since H is dense, each $U_k \subseteq G$ is dense. Hence $G - H$ is meager. But then H cannot be meager.

H is a Borel set, hence Baire measurable, hence H is open by §1.30 Corollary. Thus $H = G$. $\square$