Exercise sheet 8

Exercise 1. For $n \geqslant 2$ let $D_n := \langle a, b \mid a^2, b^2, (ab)^n \rangle$. Let $D_\infty := \langle a, b \mid a^2, b^2 \rangle$.

1. Show that in D_n the element ab has order n.
2. Show that in D_∞ the element ab has order ∞.

Hint: Consider groups satisfying these relations.

Exercise 2. Show that e is the only torsion element in the free group $F(X)$.
Hint: $F(X)$ *is residually a finite* p*-group.*

Exercise 3. Show the following isomorphisms:

1. $\varinjlim (\mathbb{Z}/2\mathbb{Z} \leftarrow \mathbb{Z} \rightarrow \mathbb{Z}/3\mathbb{Z}) \simeq \{0\}$ where the colimits are formed with respect to the canonical projections $\mathbb{Z} \rightarrow \mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$.
2. $\varinjlim (\mathbb{Z}/2\mathbb{Z} \leftarrow \mathbb{Z}/4\mathbb{Z} \rightarrow \mathbb{Z}/4\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$ where the colimits are formed with respect to the maps $\varepsilon_1 : (a + 4\mathbb{Z}) \in \mathbb{Z}/4\mathbb{Z} \mapsto (a + 2\mathbb{Z}) \in \mathbb{Z}/2\mathbb{Z}$ and $\varepsilon_2 := \mathrm{id}_{\mathbb{Z}/4\mathbb{Z}}$.

Exercise 4. Let $X \neq \emptyset$ be a set. The *support* of a permutation $\sigma \in \mathrm{Sym}(X)$ is

$$\mathrm{supp}(\sigma) := \{x \in X | \sigma(x) \neq x\}.$$

Let $\alpha, \beta, \tau, \sigma \in \mathrm{Sym}(X)$. Show the following properties:

1. If $\mathrm{supp}(\alpha) \cap \mathrm{supp}(\beta) = \emptyset$ then $[\alpha, \beta] = \mathrm{id}_X$.
2. $\alpha(\mathrm{supp}(\beta)) = \mathrm{supp}(\alpha\beta\alpha^{-1})$.
3. If $\tau(\mathrm{supp}(\alpha)) \cap (\mathrm{supp}(\alpha) \cup \mathrm{supp}(\beta)) = \emptyset$ then $[\alpha, \beta] = [[\alpha, \tau], \beta]$.
4. If $\tau(\mathrm{supp}(\alpha)) \cap (\mathrm{supp}(\alpha) \cup \mathrm{supp}(\beta)) = \emptyset$ then $[\alpha, \beta]$ is a product of at most four conjugates of τ or τ^{-1}.

Bonus exercise (“Nikolausaufgabe”)**.** Let G be a finite group.

1. Show that if G is not abelian then $[G : Z(G)] \geqslant 4$.
2. Denote by $cl(G)$ the number of conjugacy classes of G. Show that the probability p that $a, b \in G$ commute verifies

$$p(ab = ba) = \frac{cl(G)}{\#G}.$$

3. Conclude that $p(ab = ba) \leqslant 5/8$.

Please hand in your solutions on the morning of December, 9th before the lecture (letterbox 162 or electronically in the Learnweb).