Exercise sheet 6

Exercise 1. Let F_2 denote the free group on $X := \{a, b\}$.

1. Show that $\{aba^{-1}, ab\}$ is a basis for F_2.
2. Find an element $w \in F_2 \setminus \{e\}$ which is not contained in a basis of F_2.

Exercise 2. Let G be a group and let $N \subseteq G$ be a normal subgroup.
Show that G is finitely generated if both G/N and N are finitely generated.

Exercise 3 (Inner and outer automorphism)**.** Let G be a group and for $g \in G$ denote by γ_g the automorphism $\gamma_g : x \mapsto gxg^{-1}$.
The **inner automorphism group** of G is $\mathrm{Inn}(G) := \{\gamma_g \mid g \in G\}$.

1. Determine the kernel of the homomorphism
$$\Phi : \begin{cases} G & \to \mathrm{Inn}(G), \\ g & \mapsto \gamma_g. \end{cases}$$
2. Show that $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.
The quotient $\mathrm{Aut}(G)/\mathrm{Inn}(G) =: \mathrm{Out}(G)$ is the **outer automorphism group** of G.
3. Determine the group $\mathrm{Out}(\mathbb{Z}^m)$.

Exercise 4. Let $G := \langle x_n,\ n = 1, 2, 3, \dots \mid x_{n-1}^{-1} x_n^n,\ n = 1, 2, 3, \dots \rangle$. Show that $G \simeq (\mathbb{Q}, +)$.
Hint: Put $\alpha(x_n) = \frac{1}{n!}$.

Bonus exercise. Prove the following isomorphisms:

1. $\langle a, b \mid a^{-1}ba^{-2} \rangle \simeq \mathbb{Z}$.
2. $\langle a, b \mid a^5, b^3, [a, b] \rangle \simeq \mathbb{Z}/15\mathbb{Z}$.

Please hand in your solutions on the morning of November, 25th before the lecture (letterbox 162 or electronically in the Learnweb).