Exercise sheet 5

Exercise 1 (Finite index). Let G be a group with subgroups $A, B \subseteq G$. Show the following:

1. If $A \subseteq B \subseteq G$ and if both $[B : A]$ and $[G : B]$ are finite, then $[G : A] = [G : B] \cdot [B : A]$.
2. If A and B have finite index in G, then $[G : A \cap B] \leqslant [G : A] \cdot [G : B]$.

Definition 1

A subgroup H of a group G is **characteristic** if any automorphism α of G verifies $\alpha(H) = H$.

Exercise 2 (Characteristic subgroups).

1. Show that characteristic subgroups are normal.
2. Give an example of a normal subgroup which is not characteristic.

Exercise 3 (Residual finiteness criteria). Let G be a group. Show that the following are *equivalent:*

1. G is residually finite.
2. G is isomorphic to a subgroup of a product of finite groups.
3. There is a set $\mathcal{S}$ of subgroups of finite index in G such that $\cap_{H \in \mathcal{S}} H = \{e\}$.
4. There is a set $\mathcal{N}$ of normal subgroups of finite index in G such that $\cap_{N \in \mathcal{N}} N = \{e\}$.

Exercise 4 (Residual finiteness and products). Show that products of residually finite groups are again residually finite.

Bonus exercise (Non-Example). Give an example of a group which is not residually finite.

Please hand in your solutions on the morning of November, 18th before the lecture (letterbox 162 or electronically in the Learnweb).