Exercise sheet 4

Exercise 1. Let X be a finite set and let G be a finite group.

1. Determine the number of elements of Hom $(F(X), G)$.
2. Conclude that $F_l \simeq F_m$ holds for $l, m \in \mathbb{N}$ if and only if $l = m$.

Exercise 2 (Torsion elements). The elements of finite order in a group are called **torsion elements**. Let $\mathrm{Tor}(G)$ denote the set of all torsion elements in G.

1. Show that $\mathrm{Tor}(G) \subseteq G$ is a subgroup if G is abelian, and that $\mathrm{Tor}\left(G/\mathrm{Tor}(G)\right) = \{e\}$.
2. Show that $\mathrm{Tor}(G)$ need not be a subgroup if G is not abelian.
   Hint: Let $G = \{x \mapsto ax + b \mid a \neq 0,\ a, b \in \mathbb{R}\}$

Exercise 3 (Noetherian group). Let G be a group. The **commutator subgroup** of G, denoted by $[G, G]$, is is the subgroup generated by all the commutators of the group, namely

$$[G, G] := \left\langle ghg^{-1}h^{-1} \mid g, h \in G \in \right\rangle .$$

The **Heisenberg group** is defined as

$$H_3 := \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}$$

1. Determine $[H_3, H_3]$.
2. Show that all elements in $[H_3, H_3]$ commute with every element of H_3.
3. Deduce that H_3 is noetherian.

Exercise 4 (Exponent of a group). Let G be a group and denote by $o(g)$ the order of the element g in G. The **exponent** of G, denoted by $e(G)$, is

$$e(G) := \sup\{o(g) \mid g \in G\}.$$

Show that G is abelian if $e(G) = 2$. What can you say about groups with $e(G) = 3$?

Please turn the page.

Bonus exercise (Isomorphisms). Which of the following groups are isomorphic?

$$\begin{array}{ll} (\mathbb{Z},+) & (\mathbb{Z}^2,+) \\ (\mathbb{Q},+) & (\mathbb{Q}^2,+) \\ (\mathbb{R},+) & (\mathbb{R}^2,+) \end{array}$$

Hint: Every finitely generated subgroup of $\mathbb{Q}$ *is cyclic.*

Please hand in your solutions on the morning of November, 11th before the lecture (letterbox 162 or electronically in the Learnweb).