Exercise sheet 3

Exercise 1 (Isomorphisms). Let G be a group and consider two subgroups K and N where N is normal in G.

- Show that the homomorphism $G \to G/N$ induces an isomorphism:
$$K/(K \cap N) \xrightarrow{\sim} (KN)/N.$$
- Show that if K is normal in G and $N \subseteq K$, then the homomorphism $G/N \to G/K$ induces an isomorphism:
$$\frac{G/N}{K/N} \xrightarrow{\sim} G/K.$$
- Show that if K is normal in G and if $K \cap N = \{e\}$ then the following map is an isomorphism:
$$\begin{cases} K \times N & \to KN, \\ (k, n) & \mapsto kn. \end{cases}$$
- If H is another subgroup in G, show that $G = KH$ if and only if H acts transitively on G/K.

Exercise 2 (Free product). Let G and H be two non-trivial groups. Show that $G * H$ is infinite.

Exercise 3 (Hopfian groups). Show that $(\mathbb{Q}, +)$ is a Hopfian group.

Exercise 4 (Free products). Let $G = \coprod_{i \in I} G_i$ be a free product.

- Show that for every $i \in I$, there exists a homomorphism $p_i : G \to G_i$ such that $p_i \circ \iota_i = \mathrm{id}_{G_i}$.
- Show that there exists a homomorphism $\varphi : \coprod_{i \in I} G_i \to \prod_{i \in I} G_i$ such that $\mathrm{pr}_i \circ \varphi \circ \iota_i = \mathrm{id}_{G_i}$ for all $i \in I$.

Bonus exercise (Burnside's Lemma). Let G be a finite group acting on a space X. For all $g \in G$, denote $X^g := \{x \in X \mid g \cdot x = x\}$ the fixed points set of g.

1. Prove that $\sum_{x \in X} \#G_x = \sum_{g \in G} \#X^g$.
2. Show that $\sum_{x \in X} \frac{1}{\#G(x)} = \#(G \backslash X)$.
3. Deduce that
$$\#(G \backslash X) = \frac{1}{\#G} \sum_{g \in G} \#X^g.$$

Please hand in your solutions on the morning of November, 4th before the lecture (letterbox 162 or electronically in the Learnweb).