Exercise sheet 2

Exercise 1 (Generating sets and group homomorphisms). Let G be a finitely generated group and A a finite group.

1. Let S be a finite generating set of G. Show that a homomorphism from G to A is uniquely determined by its restriction to S.
2. Show that $\mathrm{Hom}(G, A)$ is a finite set.
3. Let $n > 1$. Show that G has a only a finite number of subgroups of index n.

Definition 0.1

> We say that an action of a group G on a set X is 2-transitive if G acts transitively on the set $\{(x, y) \in X^2 | x \neq y\}$, that is to say if for all $(x, y), (x', y') \in X^2$ with $x \neq y$ and $x' \neq y'$, there exists a g in G such that $g \cdot x = x'$ and $g \cdot y = y'$.

Exercise 2 (Cayley's theorem, 2-transitivity).

1. Show that every group G of order n is isomorphic to a subgroup of the symmetric group Sym(n).
   Now let G be the symmetry group of a non-square rectangle (see Figure 1).
2. Determine the order of G.
3. Let φ: $G \to$ Sym(4) be the group homorphism induced by the action of G on the corner set induced group homomorphism.
   a) What is the image of G under φ?
   b) Does G act transitively on the corners?
   c) Does G act 2-transitively?

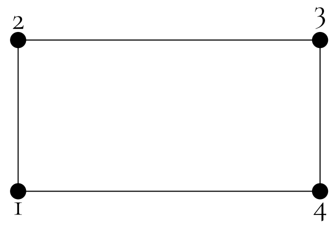

Figure 1

Please, turn the page

Exercise 3 (Rationals). Show that $(\mathbb{Q}, +)$ is not finitely generated.

Exercise 4 (Normal subgroups). Let $K, N \leqslant G$ be subgroups of G.

1. Show that if N is normal in G then KN is a subgroup of G.
2. Show that KN is normal in G if N and K are normal.
3. Let $H \leqslant G$ be a subgroup. Determine the kernel of the action

$$\begin{cases} G \times G/H & \to G/H, \\ (g, aH) & \mapsto gaH. \end{cases}$$

Bonus exercise (Double cosets). Let K, H be subgroups in G. A **double coset** is a subset of the form

$$KaH = \{kah | k \in K, h \in H\}, \quad \text{for } a \in G.$$

Show that double cosets partition G. Do all double cosets have the same number of elements?

Hint: Let K×H act on G.

Please hand in your solutions on the morning of October 28 before the lecture (letterbox 162 or electronically in the Learnweb).