Exercise sheet 10

Definition 1

> Let X be a topological space and $A \subseteq X$ a subspace. A continuous map $r : X \to X$ is called a *retraction* from X to A if $r(X) = A$ and $r(a) = a$ for all $a \in A$. A homotopy $F : X \times [0,1] \to X$, $(x,s) \mapsto F_s(x)$ between $F_0 = \mathrm{id}_X$ and a retraction F_1 from X to A is called a *deformation retract* from X to A if $F_s(a) = a$ holds for all $a \in A$ and all $s \in [0,1]$.

Exercise 1. Let X be a topological space and $A \subseteq X$ a subspace. Let $a \in A$. Denote by $\iota : A \to X$ the inclusion map. Show the following assertions:

1. If there exists a retraction r from X to A then the induced homomorphism $\iota_*\colon \pi_1(A,a) \to \pi_1(X,a)$ is injective.
2. If there exists a deformation retract from X to A, then the induced homomorphism $\iota_*\colon \pi_1(A,a) \to \pi_1(X,a)$ is an isomorphism.

Exercise 2. Calculate $\pi_1\left(\mathbb{R}^2\backslash\{0\}, (1;0)\right)$.
Hint: Show that $\mathbb{R}^2\backslash\{0\}$ *is homotopically equivalent to the* 1*-dimensional sphere* $\mathbb{S}^1$.

Exercise 3. Denote by $\|\cdot\|$ the 2-norm in $\mathbb{R}^n$. Let $D^n := \{x \in \mathbb{R}^n,\ \|x\| \leqslant 1\}$ be the n-dimensional closed unit ball and $S^n := \{x \in \mathbb{R}^n,\ \|x\| = 1\} = \partial D^{n+1}$ the n-dimensional sphere. Prove or disprove the following statements:

1. There exists a retraction $r_1 : \mathbb{R}^3 \to A$ on a subset $A \subseteq \mathbb{R}^3$ which is homeomorphic to S^1.
2. There exists a retraction $r_2 : \mathbb{R}^2 \to \{(t,0) | t \in \mathbb{R}\}$.
3. Let X be a topological space and $Y \subseteq X$ a subspace and $y \in Y$. If $\pi_1(X,y)$ is free and $\pi_1(Y,y) \neq \{1\}$ is finite then there exists no retraction $r_3 : X \to Y$.
4. There exists a retraction $r_4 : S^1 \times D^2 \to S^1 \times S^1$ where $S^1 \times S^1$ is the boundary of the solid torus $S^1 \times D^2$, namely $S^1 \times S^1 = \partial\left(S^1 \times D^2\right)$.

Exercise 4. Let G be a countable group. Show that there exists a group G^* which contains G as a subgroup and in which all elements of the same order are conjugate to each other.

Bonus exercise. Let G be a finitely generated group. Show that the following are equivalent:

1. G contains uncountably many normal subgroups.
2. G has uncountably many, pairwise non-isomorphic quotients.

Please hand in your solutions on the morning of December, 23rd before the lecture (letterbox 162 or electronically in the Learnweb).