Exercise sheet 1

Exercise 1 (Abelian group). Let G be a group. Show that G is abelian if and only if the following map is a group homomorphism

$$\iota : \begin{cases} G & \to G, \\ g & \mapsto g^{-1}. \end{cases}$$

Exercise 2 (Subgroups). Let G be a group and A and B two subgroups of G. Recall that $A \cdot B$ is defined as $\{ab | a \in A, b \in B\}$.

1. Show that $A \cdot B$ is a subgroup of G of and only if $A \cdot B = B \cdot A$.
2. Find a group G and two subgroups A and B such that $A \cdot B$ is not a subgroup of G.

Exercise 3 (Famous subgroups). Let G be a group.

1. The *center* of G is defined as $Z(G) := \{z \in G | \forall g \in G, zg = gz\}$. Show that $Z(G)$ is a subgroup of G.
2. Let $H \leqslant G$. The *normalizer* of H in G is defined as $N_G(H) := \{g \in G | gHg^{-1} = H\}$. Show that $N_G(H)$ is the largest subgroup of G in which H is normal.

Exercise 4 (Group action). Let $n \geqslant 1$. Let $X := \mathbb{R}^n$ and let $(e_1, \dots, e_n)$ be its standard basis. Show that the following map defines an *action* of Sym(n) on X:

$$\begin{cases} \mathrm{Sym}(n) \times X & \to X, \\ (\sigma, \sum_{i=1}^n x_i e_i) & \mapsto \sum_{i=1}^n x_i e_{\sigma(i)}. \end{cases}$$

Bonus exercise (Group homomorphism).

1. Let G and H be two groups and let $\varphi : G \to H$ be a group homomorphism. Show that for all $g \in G$ we have $\mathrm{ord}(\varphi(g)) | \mathrm{ord}(g)$.
2. Determine all group homorphisms $\varphi : \mathbb{Z}/4\mathbb{Z} \to \mathrm{Sym}(3)$ and $\psi : \mathrm{Sym}(3) \to \mathbb{Z}/4\mathbb{Z}$.

Please hand in your solutions on the morning of October 21st before the lecture (letterbox 162 or electronically in the Learnweb).