§7 Graphs and subgroups of free groups

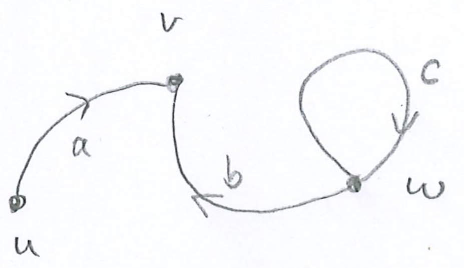
1\. Def A Serre graph Γ consists of the following data: a set of vertices V, a set of edges E and three maps

$$e \mapsto \bar{e}, \quad E \to E \qquad \text{(edge inversion)}$$
$$e \mapsto e_0, \quad E \to V \qquad \text{(starting point)}$$
$$e \mapsto e_1, \quad E \to V \qquad \text{(end point)}$$

such that $\bar{e} \neq e$, $\bar{\bar{e}} = e$, $\bar{e}_0 = e_1$, $\bar{e}_1 = e_0$.

An orientation of Γ is a subset $E^+ \subseteq E$ such that $e \in E^+ \Leftrightarrow \bar{e} \notin E^+$ holds.

We picture the orientation using arrows



$V = \{u, v, w\}$

$E = \{a, b, c, \bar{a}, \bar{b}, \bar{c}\}$

$E^+ = \{a, b, c\}$

A subgraph is defined in the obvious way,

$V' \subseteq V$

$E' \subseteq E$

A <u>morphism</u> of graphs is a map $\varphi : \Gamma_1 \to \Gamma_2$,

$\varphi : V_1 \to V_2$ with $\varphi(e)_0 = \varphi(e_0)$

$\varphi : E_1 \to E_2$ $\varphi(e)_1 = \varphi(e_1)$

$\varphi(\bar{e}) = \overline{\varphi(e)}$.

The <u>geometric realization</u> $|\Gamma|$ of a Serre graph Γ is defined as follows. Let $E^+ \subseteq E$ be an orientation, put

$$|\Gamma| = V \dot\cup (E^+ \times [0,1]) / \sim$$

where $\sim$ is the equivalence relation

$x \sim x$ and $(e,0) \sim e_0$ for $e \in E^+$

$x \in V \cup (E \cup [0,1])$ $(e,1) \sim e_1$

$\uparrow$ glue $e \times [0,1]$ to e_0, e_1

We endow $|\Gamma|$ with the quotient topology for the map $V \dot\cup (E^+ \times [0,1]) \xrightarrow{q} |\Gamma|$

($W \subseteq |\Gamma|$ is open iff $q^{-1}(W)$ is open).

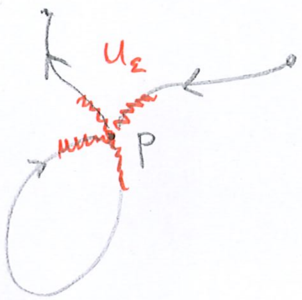
For $(e,s) \in E^+ \times [0,1]$ /~ we write

$q(e,s) = e_s$ for short.

What do open neighbourhoods in $|\Gamma|$ look like?

Let $p \in |\Gamma|$

(1) If $p = q(v)$, $v \in V$ is a vertex, let $0 < \varepsilon < \frac{1}{2}$ and put $U_\varepsilon(p) = p \cup \{e_s \mid e_0 = p,\ 0 \le s < \varepsilon, e \in E^+\}$
$\cup \{e_s \mid e_1 = p,\ 1-\varepsilon < s \le 1,\ e \in E^+\}$

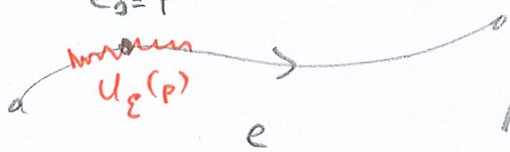


⌈Strictly speaking we have to allow ε to vary with the edges...⌋

(2) If $p = e_s$, $s \neq 0,1$ is not a vertex, let $\varepsilon > 0$ with

$0 < s-\varepsilon < s+\varepsilon < 1$

and put $U_\varepsilon(p) = \{e_t \mid s-\varepsilon < t < s+\varepsilon\}$



Note: these $U_\varepsilon(p)$ are contractible.

We note the following.

(a) $|\Gamma|$ is a Hausdorff space (use (1), (2)).

(b) $|\Gamma|$ is locally path connected

(c) $|\Gamma|$ is compact iff V, E are finite (!)

(d) $|\Gamma|$ is independent of the choice of $E^+ \subseteq E$.

put $|\Gamma| = V \dot\cup (E \times [0,1])/\sim$ with the additional relation $(e,s) \sim (\bar{e}, 1-s)$

We put $e_s = (e,s)/\sim$ for $e \in E$.

(e) If Γ' is a subgraph of Γ, then $|\Gamma'| \subseteq |\Gamma|$ is closed

(f) If $\varphi : \Gamma_1 \to \Gamma_2$ is a morphism, then φ induces a continuous map $\varphi : |\Gamma_1| \to |\Gamma_2|$ via $\varphi(e_s) = \varphi(e)_s$,

2. <u>Edge paths</u> For every edge, $t \mapsto e_t$ is a path in $|\Gamma|$ from e_0 to e_1. A concatenation of such paths is called an <u>edge path</u>. The <u>length</u> of the edge path is the number of edges used.

<u>Lemma</u> Let u, v be vertices in $|\Gamma|$ and $\alpha \in \Omega(|\Gamma|; u, v)$. Then there is an edge path γ with $\alpha \simeq \gamma$ rel ∂.

<u>Pf</u> We cover $|\Gamma|$ by the open sets $U_{\frac{1}{2}}(p)$, p vertex and $U_{\frac{1}{2}}(e_{\frac{1}{2}})$ $e \in E$

Since $[0,1]$ is compact, there are $s_0 = 0 < s_1 < \dots < s_m = 1$

such that $\alpha([s_{k-1}, s_k]) \in U_{\frac{1}{2}}(z_k)$ as above

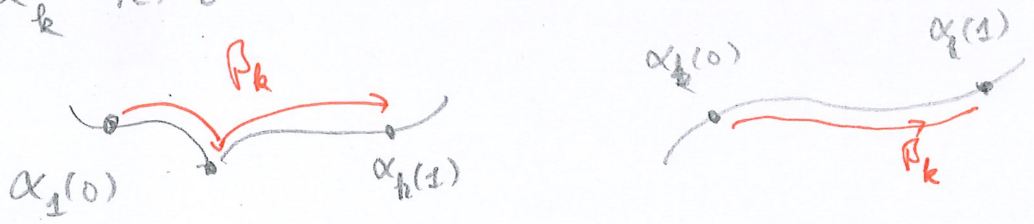
(z_k vertex or $z_k = e_{\frac{1}{2}}$)

put $\alpha_k = \alpha(s_{k-1} + t(s_k - s_{k-1})) \rightsquigarrow$

$\alpha \simeq \alpha_m * (\alpha_{m-1} * (\dots \alpha_1)))$ rel. ∂

For each k, we can straighten α_k to

$\beta_k \simeq \alpha_k$ rel ∂



because the $U_{\frac{1}{2}}(z_k)$ are contractible

($\Rightarrow$ all path in $U_{\frac{1}{2}}(z_k)$ between two points are homotopic rel ∂)

Hence $\alpha \simeq \alpha_m * (\alpha_{m-1} * \dots * \alpha_1)) \simeq \beta_m * (\beta_{m-1} * (\dots * \beta_1))) = \beta$

and β is homotopic to an edge path rel ∂.

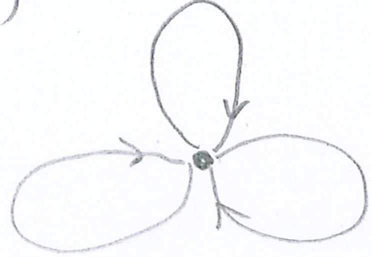
$\square$

We call an edge path of minimal length between two vertices <u>minimal</u>. We call it <u>reduced</u> if it does not have parts of the form $\gamma_{\bar{e}} * \gamma_e$ $\quad \gamma_e(t) = e_t$

(no <u>backtracking</u>)

Since $\gamma_{\bar{e}} * \gamma_e \simeq \varepsilon_{e_0}$, a minimal edge path is reduced.

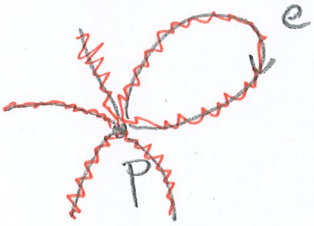
3. <u>The Rose</u> A graph with one vertex p is called a rose (with n petals if $\# E = 2n$ is finite)



<u>Proposition</u> Let Γ be a rose, let E^+ be an orientation. Then $\pi_1(|\Gamma|; p)$ is a free group, with basis $\{ [\gamma_e] \mid e \in E^+ \}$

$$\gamma_e(t) = e_t$$

<u>pf</u> This is true if $E = \phi$, so assume $E^+ \neq \phi$. For each $e \in E^+$, put $W_e = U_{\frac{1}{2}}(p) \cup \{ e_s \mid 0 \le s \le 1 \}$



and $\mathcal{U} = \{ W_e \mid e \in E^+ \}$

If $a \neq b$, $a, b \in E^+$, then

$W_a \cap W_b = U_{\frac{1}{2}}(p)$ is contractible.

Also, $W_a \cap W_b \cap W_c$ is path connected for all $a, b, c \in E^+$

Hence $\pi_1(|\Gamma|; p) \cong \coprod_{a \in E^+} \pi_1(W_a; p)$

by § 6.20 Cor A.

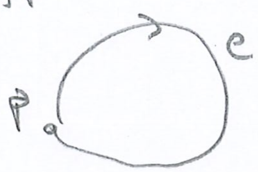
Put $A_e = \{e_s \mid 0 \leq s \leq 1\} \subseteq W_e$. There is a strong deformation retraction $W_e \to A_e$ as follows:

$$h_s(a_t) = \begin{cases} a_t & \text{if } a = e \\ a_{t(1-s)} & \text{if } a \neq e,\ 0 \leq t < \frac{1}{2} \\ a_{t+s(1-t)} & \text{if } a \neq e,\ \frac{1}{2} < t \leq 1 \end{cases}$$

$h_0 = \mathrm{id}_{W_e}$, $h_s(a_t) = a_t$ for $a_t \in A_e$, $h_1(W_e) \subseteq A_e$

Hence $A_e \hookrightarrow W_e$ induces an isomorphism

$\pi_1(A_e; p) \xrightarrow{\cong} \pi_1(W_e; p)$

Since $A_e \cong S^1$ 

,

$\pi_1(A_e; p) \cong \mathbb{Z}$ by § 6.10 Cor D, with generator $[\gamma_e]$, the claim follows. $\square$

Our next aim is to show that every connected graph is homotopy equivalent to some rose. This will show that fundamental groups of graphs are always free.

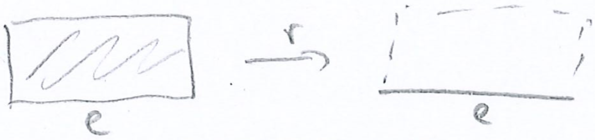
4. <u>Def</u> Let X be a topological space (Hausdorff, as usual...), let $A \subseteq X$ be closed. We say that $A \subseteq X$ has the <u>homotopy extension property</u> HEP if there is a retraction

$$r : X \times [0,1] \longrightarrow (X \times \{0\}) \cup (A \times [0,1])$$

<u>Fact 1</u> If Γ' is a subgraph of Γ, then $|\Gamma'| \subseteq |\Gamma|$ has the HEP.

<u>pf</u> If e is an edge in Γ which is not in Γ', define r on $e \times [0,1]$ as follows

(a) $e_0, e_1 \notin \Gamma' \Rightarrow r(e_s, t) = (e_s, 0)$



(b) $e_0, e_1 \in \Gamma'$

(c) $e_0 \in \Gamma'$
$e_1 \notin \Gamma'$

(d) $e_0 \notin \Gamma'$
$e_1 \in \Gamma$

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<u>Fact 2</u> Suppose that $A \subseteq X$ has the HEP. Let X/A denote the space obtained by collapsing A to a single point ($X/A = X/\sim$, with $x \sim x$ for $x \in X$ and $a \sim b$ for $a, b \in A$) endowed with the quotient topology for the collapsing map $q : X \to X/A$. If A is contractible, then q is a homotopy equivalence. Hence for $p \in A$, the map

$q_* : \pi_1(X; p) \to \pi_1(X/A; q(p))$ is an isomorphism.

Pf Let $h : A \times [0,1] \to A$ be a map contracting A to a point, $h_0 = id_A$. Using the HEP, we extend h to a map $h : X \times [0,1] \to X$ with $h_0 = id_X$. Consider for $s \in [0,1]$

$$\begin{tikzcd} X \arrow[r, "h_s"] \arrow[d, "q"'] & X \arrow[d, "q"] \\ X/A \arrow[r, "\bar{h}_s"] & X/A \end{tikzcd}$$

One shows that $\bar{h} : X/A \times [0,1] \to X/A$ is continuous (use Whitehead's Lemma!)

For $s=1$ we have

$$\begin{tikzcd} X \arrow[r, "h_1"] \arrow[d, "q"'] & X \arrow[d] \\ X/A \arrow[r, "\bar{h}_1"] \arrow[ur, "f"] & X/A \end{tikzcd}$$

$h_1(A) = \{a\} \subseteq A$

$\Rightarrow$ f is continuous

hence

$h_1 = f \circ q \simeq h_0 = id_X$

$q \circ f = \bar{h}_1 \simeq \bar{h}_0 = id_{X/A}$

$\square$

5. For a graph Γ, the following are equivalent:

(1) $|\Gamma|$ is connected

(2) $|\Gamma|$ is path connected (because $|\Gamma|$ is locally path connected)

(3) Any two edges in Γ can be joined by an edge path (use §7.2)

<u>Theorem</u> Let Γ be a connected graph, with a subgraph $\Gamma' \subseteq \Gamma$. Then there is a subgraph Δ, with $\Gamma' \subseteq \Delta \subseteq \Gamma$ and the following hold

(1) Δ contains all vertices of Γ

(2) $|\Gamma'| \subseteq |\Delta|$ is a strong deformation retract.

(hence $|\Gamma'| \simeq |\Delta|$)

pf Put $\Delta_0 = \Gamma'$ and construct Δ_{k+1} from Δ_k as follows. If a vertex v of Γ is not in Δ_k, but connected by an edge e to a vertex u in Δ_k, add e to Δ_k. Do this for all vertices; the result is Δ_{k+1}.

Put $\Delta = \bigcup_{k\geq 0} \Delta_k$. Then Δ contains all vertices of Γ, since Γ is connected.

Moreover, $|\Delta_k| \subseteq |\Delta_{k+1}|$ is a strong deformation retract. Put $s_k = \sum_{j=1}^{k} 2^{-j}$. We can choose a homotopy $h^k : |\Delta_{k+1}| \times [0,1] \to |\Delta_{k+1}|$ with $h^k_s = id$ for $0 \leq s \leq s_k$

$h^k_s = h^k_1$ for $s_{k+1} \leq s \leq 1$

in the sub-interval $s_k \leq s \leq s_{k+1}$, h^k retracts $|\Delta_{k+1}|$ to $|\Delta_k|$.

Then $H^k_s = h^0_s \circ h^1_s \circ \ldots \circ h^k_s$

retract $|\Delta_{k+1}|$ to $|\Delta_0|$ and H^l_s extends H^k_s for $l \geq k$.

Put $H_s = \bigcup_{k \geq 0} H_s^k$, this is a well-defined continuous map $H_s : |\Delta| \times [0,1] \to |\Delta|$ that retracts $|\Delta|$ to $|\Delta_0| = |\Gamma'|$ $\square$

Corollary A If $|\Gamma|$ is connected, then there is a subgraph $\Delta \subseteq \Gamma$ containing all vertices of Γ, such that $|\Delta|$ is contractible.

pf let v be any vertex in Γ and put $\Gamma' = \{v\}$. $\square$

(Such a subgraph is called a spanning tree)

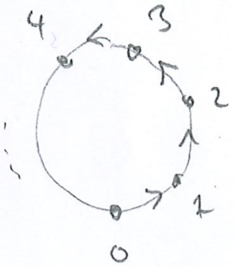
Corollary B Let Γ be a connected graph and $\Gamma' \subseteq \Gamma$ a connected subgraph. Then there is a retraction $r : |\Gamma| \to |\Gamma'|$.

pf Choose an orientation E^+ for Γ, and Δ as in the theorem. There is a retraction $r_1 : |\Delta| \to |\Gamma'|$ by the theorem.

We define a retraction $r_2 : |\Gamma| \to |\Delta|$ as follows. If e_s is on an edge $e \in$ of Γ', put $r_2(e_s) = e_s$.

If $e \in E^+$ is not in Δ', then e_0, e_1 are in Δ', hence there is a path α in $|\Delta|$ from e_0 to e_1. Put $r_2(e_s) = \alpha(s)$. Then $r_2 : |\Gamma| \to |\Delta|$ is a continuous retraction, put $r = r_1 \circ r_2 : |\Gamma| \to |\Gamma'|$ □

6. <u>Def</u> Let $m \geq 1$ and $V = \{0, 1, \dots, m-1\}$. Let C_m denote the graph with an oriented edge e_k from $k-1$ to k, and e_m from $m-1$ to 0. We call C_m an <u>m-circle graph</u>.



Note $|C_m| \cong S^1$

A connected graph T is called a <u>tree</u> if it contains no subgraph isomorphic to a circle graph C_m, for $m \geq 1$.

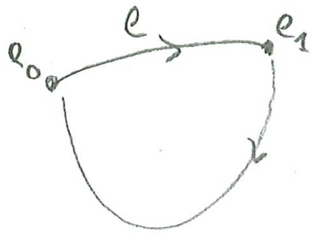
<u>Theorem</u> Let Γ be a graph. The following are equivalent.

(i) Γ is a tree

(ii) $|\Gamma|$ is contractible

(iii) $|\Gamma|$ is 1-connected (= path connected with trivial fundamental group).

Pf (i) $\Rightarrow$ (ii): Choose a vertex $v \in \Gamma$, put $\Gamma' = \{v\} \subseteq \Gamma$ and Δ as in Thm §7.5.
Thus $|\Delta|$ is contractible. <u>We claim $\Delta = \Gamma$.</u>
If e is an edge in Γ which is not in Δ, then there is a minimal edge path α in Δ connecting e_1 and e_0



, this yields a circle graph in Γ ↯ $\square$

(ii) $\Rightarrow$ (iii) is true in general.

(iii) $\Rightarrow$ (i) show <u>$\neg$(i) $\Rightarrow$ $\neg$(iii).</u> (Clear if Γ is not connected.)

Let $\Gamma' \subseteq \Gamma$ be a circle graph. By §7.5 Cor B there is a retraction $r : |\Gamma| \to |\Gamma'|$, consider for $p \in \Gamma'$

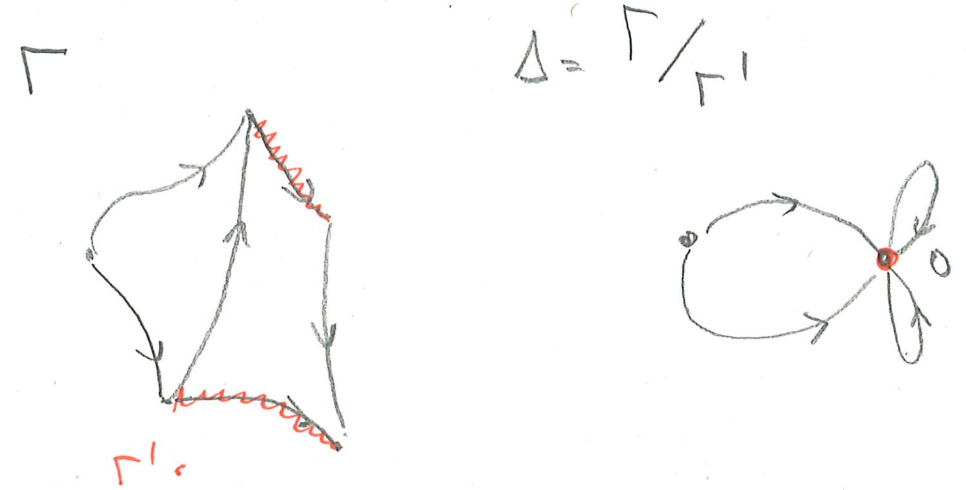
$$\begin{tikzcd} {|\Gamma'|} \arrow[r, hook, "i"] \arrow[dr, equal] & {|\Gamma|} \arrow[d, "r"] \\ & {|\Gamma'|} \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \pi_1(|\Gamma'|;p) \arrow[r, "i_*"] \arrow[dr, equal] & \pi_1(|\Gamma|;p) \arrow[d, "r_*"] \\ & \underbrace{\pi_1(|\Gamma'|;p)}_{\cong \mathbb{Z}} \end{tikzcd}$$

hence $\pi_1(|\Gamma|;p)$ is nontrivial $\square$

<u>Corollary C</u> (to Thm §7.5)

If Γ is a connected graph, then there is a subtree $T \subseteq \Gamma$ containing all vertices of Γ, a <u>spanning tree</u>.

7. <u>Construction</u> Let $\Gamma' \subseteq \Gamma$ be a subgraph, $\Gamma' \neq \emptyset$. Let $\Delta = \Gamma/_{\Gamma'}$ denote the following graph. The vertex set of Δ is the set of vertices v which are <u>not</u> in Γ' and one new vertex o. The edges of Δ are the edges not in Γ'. If such an edge e has $e_0 \in \Gamma'$, let e in Δ start in o, similarly if $e_1 \in \Gamma'$.



Then $|\Delta| \cong |\Gamma|/_{|\Gamma'|}$.

8. <u>Theorem</u> Let Γ be a connected graph, let $T \subseteq \Gamma$ be a spanning tree, let E^+ be an orientation for Γ. Let $X \subseteq E^+$ denote all edges which are <u>not</u> in T, let $p \in T$ be a vertex.

Then there are reduced edge paths

$\alpha_e \in \Omega(|\Gamma|;p)$, for $e \in X$ such that

$\pi_1(|\Gamma|;p)$ is a free group with basis

$\{\alpha_e \mid e \in X\}$

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pf Put $\Delta = \Gamma/T$, then $|\Delta| \cong |\Gamma|/|T|$

Since $|T|$ is contractible and $|T| \subseteq |\Gamma|$ has the HEP, $q: |\Gamma| \to |\Delta|$ is a homotopy equivalence (§7.4). Moreover, Δ is a rose (because Δ contains all vertices of Γ), hence $\pi_1(|\Delta|;o)$ is a free grp, with basis $\{q_*[\beta_e] \mid e \in X\}$

$\beta_e(t) = q(e_t)$. By §7.2, there is an edge path $\alpha \in \Omega(|\Gamma|;p)$ with $q_*[\alpha] = [\beta_e]$. Among all such α, choose one of minimal length and call it α_e. Then α_e is reduced, and $q_*[\alpha_e] = [\beta_e]$.

$\square$

Corollary A Every connected graph is homotopy equivalent to some rose.

Corollary B If T is a spanning tree in a graph Γ, with orientation E^+, then the number of the edges $\{a \in E^+ \mid a \text{ not in } T\}$ is independent of T.

Corollary C Let Γ be a connected graph, let u, v be vertices, with $\alpha \in \Omega(|\Gamma|; u, v)$. Then there is a unique reduced edge path γ with $\alpha \simeq \gamma$ rel ∂.

Pf (*) Let V denote the vertex set of Γ, E^+ an orientation

consider $q : |\Gamma| \to |\Gamma/_V|$.

If $\beta \simeq \gamma$ rel ∂, then $[q\circ\beta] = [q\circ\gamma]$ in $\pi_1(|\Gamma/_V|; p)$

The edges in γ form a reduced word in the free group $\pi_1(|\Gamma/_V|; p)$ for the basis consisting of the edges in E^+. Hence we can read off the edges in γ from $[q\circ\alpha]$.

(*) The existence of γ was shown in §7.2

$\square$

9. Def Let G be a grp and $S \subseteq G$ a subset. The <u>oriented Cayley graph</u> $C(G,S)$ is the (oriented) graph with vertex set $V = G$ and $E^+ = \{(g,gs) \mid g \in G, s \in S\}$

$$(g,gs)_0 = g$$

$$(g,gs)_1 = gs$$

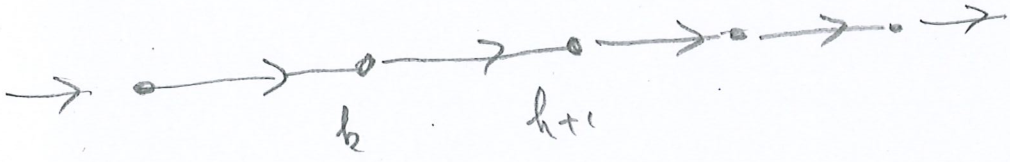
For each edge (g,gs) we define formally the inverse

$$\overline{(g,gs)} \notin E^+, \quad \overline{\overline{(g,gs)}} = (g,gs)$$

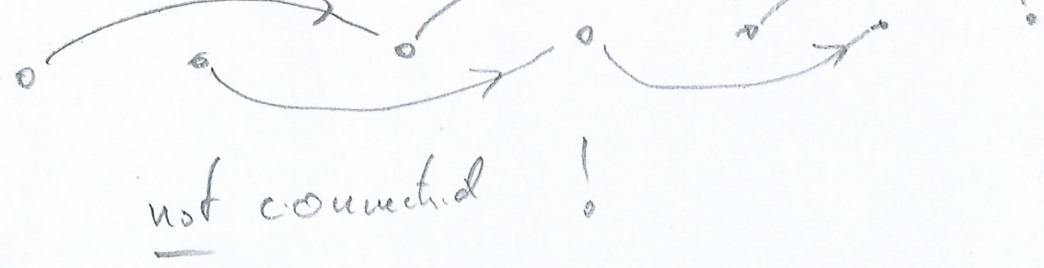
$$E = E^+ \,\dot{\cup}\, \{\overline{(g,gs)} \mid g \in G, s \in S\}.$$

<u>Examples</u>

(a) $G = \mathbb{Z}$, $S = \{1\}$



(b) $G = \mathbb{Z}$, $S = \{2\}$



<u>not</u> connected !

(c) $G = \mathbb{Z}/m$, $m \geq 2$ $\quad S = \{1\}$

$C(G,S) \cong C_m$

Observations about oriented Cayley graphs.

(1) G acts on $C(G,S)$ via

$$g(a, as) = (ga, gas)$$

and preserves the orientation. The action on V is transitive and free, the action on E^+ (or E) is free.

(2) $a, b \in V$ are connected by an edge in E^+

$a \to b$ iff $a^{-1}b \in S$

(3) The subgraph  is in $C(G,S)$ iff $e \in S$ (e = unit element in G)

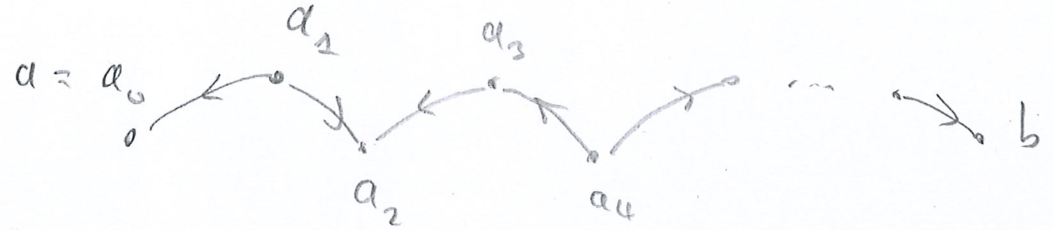
(4) The subgraph  cannot occur. (edges in E^+)

The subgraph  occurs iff there is $s \in S$ with $s \neq e$ and $s^{-1} \in S$

(edges in E^+)

(5) S generates G iff $C(G,S)$ is connected.

pf There is an edge path in $C(G,S)$ from a to b iff there are $a_0 = a, a_1, \dots, a_m = b$

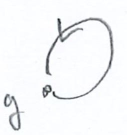


with $a_k^{-1} a_{k+1} \in S \cup S^{-1}$, or $b \in a\langle S \rangle$

This holds (for all $a, b \in G$) iff S generates G. □

10. **Lemma** Let $G = F(X)$ be a free group. Then $C(G,X)$ is a tree. Each vertex g is the starting point of $\#X$ oriented edges.

pf $\Gamma = C(G,X)$ is connected, since X generates G. The vertex g is the starting point of the oriented edges $\{(g, gx) \mid x \in X\}$. Since $e \notin X$ and $x^{-1} \notin X$ for all $x \in X$, Γ contains no edges



or

. Suppose that Γ has a subgraph $\Gamma' \cong C_m$, for $m \geq 3$. Thus we have

$g_0, g_1, \dots, g_{m-1}, g_m = g_0$ in G

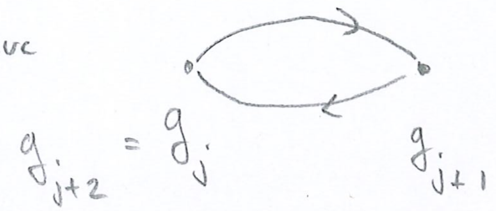
with $\quad g_{k+1} = g_k x_k^{\varepsilon_k} \qquad \varepsilon_k = \pm 1$

$\qquad\qquad\qquad\qquad\qquad\qquad x_k \in X$

$$g_0 = g_m = g_0 x_0^{\varepsilon_0} \cdots x_{m-1}^{\varepsilon_{m-1}} \Rightarrow x_0^{\varepsilon_0} \cdots x_{m-1}^{\varepsilon_{m-1}} = e$$

Hence there is j with $x_j = x_{j+1}$ and $\varepsilon_j = -\varepsilon_{j+1}$

$\rightsquigarrow$ in Γ' we have



$\lightning \qquad \square$

11. <u>Def</u> Let G be a grp acting on a graph Γ.
We say that the action is <u>without inversion</u> if there is no edge e with $g(e) = \bar{e}$.

G acts on the Cayley graph $\mathcal{G}(G,S)$ without inversion.

<u>Lemma</u> If G acts (freely and!) without inversion on a graph Γ, then the action $G \times |\Gamma| \to |\Gamma|$ is a covering space action (cp. §6.16).

We define a graph $_G\backslash\Gamma$ with vertex set $\tilde{V}$ and edge set $\tilde{E}$ as follows, for $\Gamma = (V,E)$

$$\tilde{V} = {}_G\backslash V = \{ G(v) \mid v \in V \}$$
$$\tilde{E} = {}_G\backslash E = \{ G(e) \mid e \in E \}$$

$G(e)_0 = G(e_0)$

$G(e)_1 = G(e_1)$.

Then $|{}_G\backslash\Gamma| \cong {}_G\backslash|\Gamma|$

pf The neighborhoods $U_\varepsilon(p)$ defined in §7.1 for $\varepsilon > 0$ small have the property required for a covering space action. (because the action is free!) The rest follows from the definitions. (!) $\square$

12. Theorem Let G be a group. The following are equivalent. (i) $G \cong F(X)$ is a free group

(ii) G acts freely and without inversion on a tree T.

pf (i) ⇒ (ii) by §7.11.

(ii) ⇒ (i): Put $\Gamma = {}_G\backslash T$. Then

$G \cong \pi_1(|\Gamma|; p)$ by §6.17, since

$\pi_1(|T|; p) = \{e\}$ is the trivial grp.

But $\pi_1(|\Gamma|; p)$ is a free group by §7.8.

Corollary (Schreier's Theorem)

If $H \leq F(X)$ is a subgroup, then H is a free group.

pf H acts freely and without inversion on the tree $T = C(F(X), X)$. $\square$

The remainder of this chapter is based on Stallings' paper "Topology of finite graphs". #

13. Def A morphism of graphs $\varphi : \Gamma_1 \to \Gamma_2$ is called an immersion if for every vertex v in Γ_1, the star $\Gamma_1 = (V_1, E_1)$

$$st(\Gamma_1, v) = \{ a \in E_1 \mid a_0 = v \}$$

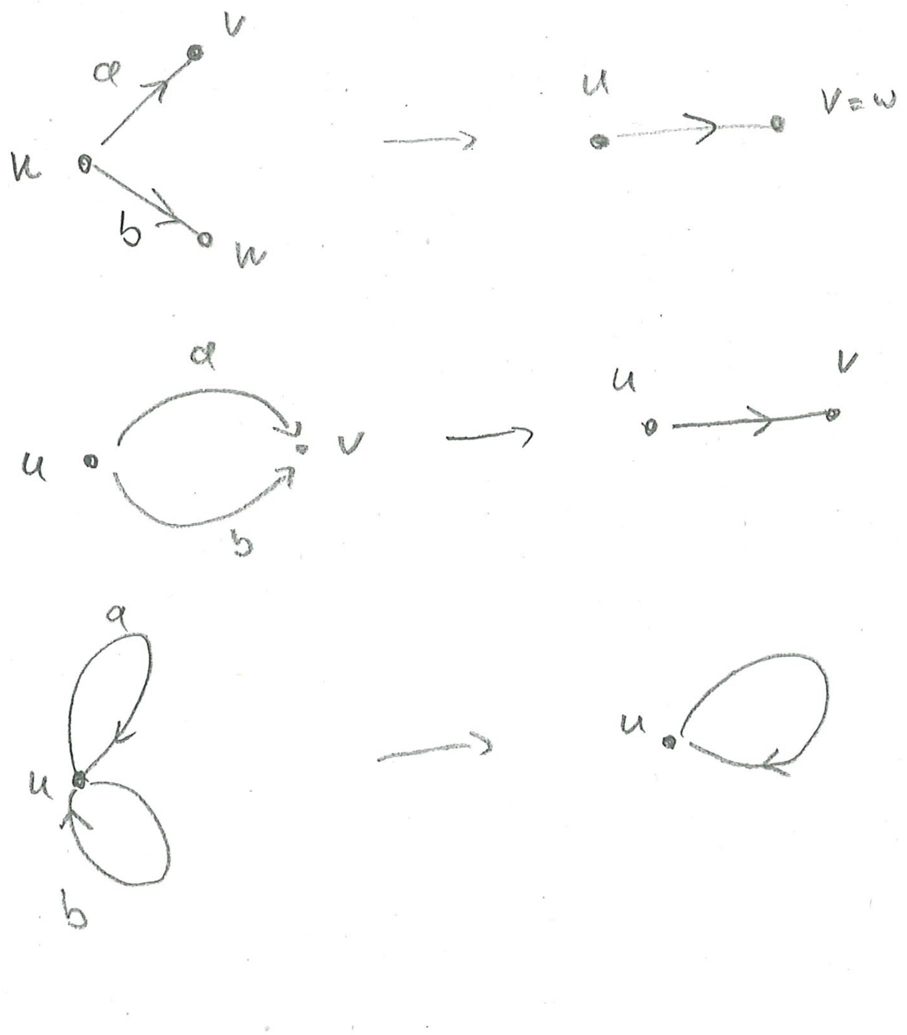
is mapped injectively to $st(\Gamma_2, \varphi(v))$.

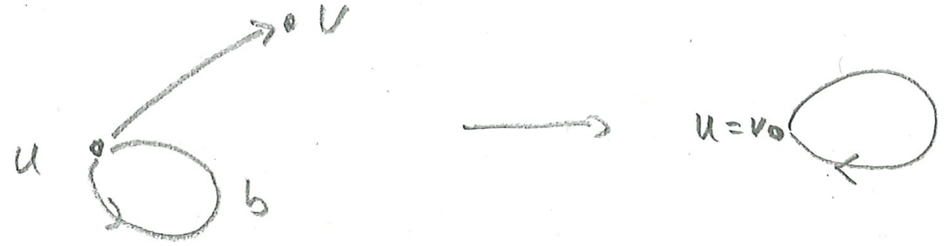
Equivalently, φ is injective on $U_{\frac{1}{2}}(v) \subseteq |\Gamma_1|$.

14. Lemma Let $\varphi : \Gamma_1 \to \Gamma_2$ be an immersion, let $v \in V_1$. Then $\varphi_* : \pi_1(|\Gamma_1|, v) \to \pi_1(|\Gamma_2|, \varphi(v))$ is injective.

pf An immersion maps reduced paths to reduced path. Hence the claim follows from §7.8 Cor. C. $\square$

15. Def (Foldings) Let a, b be edges in a graph Γ, with $a_0 = b_0$ and $a \neq \bar{b}$. The graph $\Gamma/_{a=b}$ is the graph obtained by identifying a and b,





Note that there is a morphism $q : \Gamma \to \Gamma/_{a=b}$

Observations

(1) Every edge path starting and ending in $u = a_0$ in $\Gamma/_{a=b}$ has a preimage in Γ. Hence $\pi_1(|\Gamma|; u) \to \pi_1(|\Gamma/_{a=b}|, u)$ is surjective.

(2) If a morphism $\varphi : \Gamma_1 \to \Gamma_2$ is <u>not</u> an immersion, then there are edges $a, b \in E_1$ with $a_0 = b_0$, $a \neq \bar{b}$, $\varphi(a) = \varphi(b)$, hence φ factors as

$$\begin{tikzcd} \Gamma_1 \arrow[rr, "\varphi"] \arrow[dr, "q"'] & & \Gamma_2 \\ & \Gamma_1/_{a=b} \arrow[ur, "\varphi'"'] & \end{tikzcd}$$

(3) If $\Gamma' \xrightarrow{\varphi} \Gamma$ is a morphism of graphs and if Γ' is finite, then φ factors as

$$\Gamma' = \Gamma_1 \to \Gamma_2 \to \Gamma_3 \to \dots \to \Gamma_m \to \Gamma$$

such that $\Gamma_k \to \Gamma_{k+1}$ is a folding, and $\Gamma_m \to \Gamma$ is an immersion.

16. An algorithm

Proposition Let Γ be a connected graph, p a vertex, let $\alpha_1, \dots, \alpha_n$ be reduced edge paths in $\Omega(|\Gamma|;p)$, let

$$H = \langle [\alpha_1], \dots, [\alpha_n] \rangle \leq \pi_1(|\Gamma|;p).$$

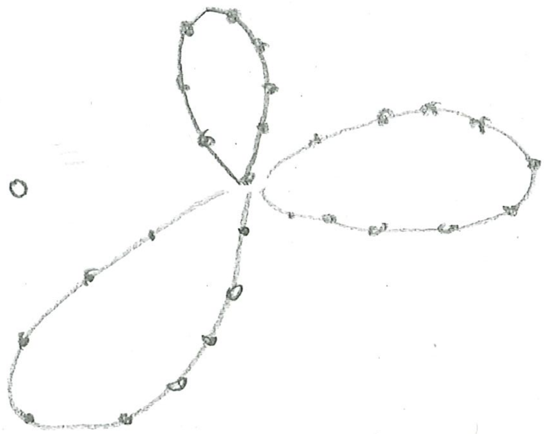
Then there is a finite connected graph Δ, and an immersion $\varphi : \Delta \to \Gamma$ such that $H = \varphi_* \pi_1(|\Delta|;x)$ for some vertex x in Δ.

Pf Let Δ_0 denote the graph consisting of n circles of length r_k,

r_k = length of the edge path α_k

with one vertex x in common.

Δ_0



Let $\varphi : \Delta_0 \to \Gamma$ denote the morphism which maps the kth circle to α_k.

Now apply observation (3)

$$\Delta_0 \to \Delta_1 \to \Delta_2 \to \cdots \to \Delta_m \xrightarrow{\varphi} \Gamma$$

surjective on π_1

immersion

put $\Delta = \Delta_m$

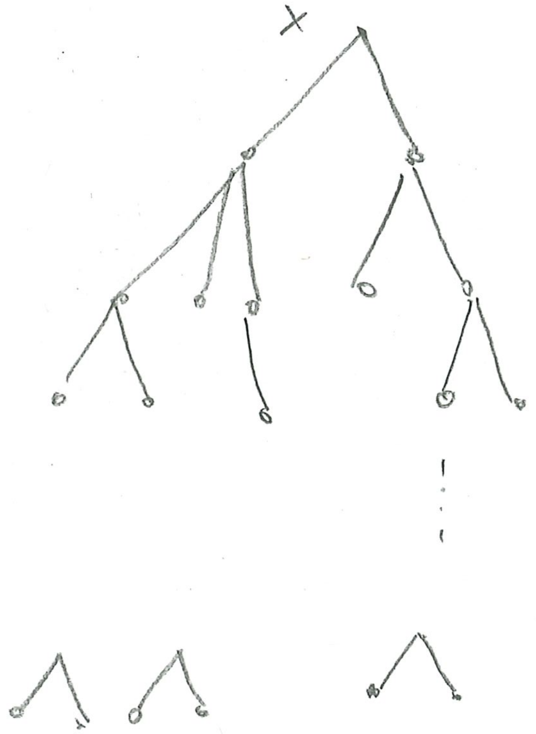
$$\varphi_* \pi_1(|\Delta_0|; x) = H = \varphi_* \pi_1(|\Delta_m|; x) \qquad \square$$

Remark If we choose a spanning tree in Δ, we obtain a $\underbrace{\ }_{\text{(finite)}}$ basis for the free group H.

Corollary If $H \leq F(X)$ is a finitely generated subgroup, then there is a finite subset $Y \subseteq H$ which is a basis for the free group H, $H \cong F(Y)$.

17. **Lemma** Let T be a finite tree with ℓ vertices. Then $\# E^+ = \ell - 1$ (E^+ any orientation)

pf choose any vertex x, and count



vertices	edges
1	
	n_1
n_1	
	n_3
n_3	
	n_4
n_4	
$\vdots$	$\vdots$
	n_r
n_r	

$$\#\text{vertices} = 1 + n_1 + n_2 + \dots + n_r = \ell$$

$$\#\text{edges} = n_1 + n_2 + \dots + n_r$$

$\square$

Corollary If Γ is a finite connected graph with ℓ vertices and $\# E^+ = m$, then $\pi_1(|\Gamma|; p) \cong F_{m-\ell+1}$ $\square$

18. <u>Proposition</u> Let $H \subseteq F_m$ be a subgrp of finite index $[F_m : H] = n$. Then

$$H \cong F_\ell \,, \quad \ell = n(m-1)+1.$$

<u>pf</u> Let $X \subseteq F_m$ be a basis, $\#X = m$, put $T = C(F_m, X)$ and $\Delta = {}_H\backslash T$. Then $\pi_1(|\Delta|; p) \cong H$ by §6.17. Now we count.

vertices in Δ are H-orbits in G $\leadsto$ n vertices.

edges in Δ are H-orbits in E^+

$$g(a, ax) = (b, by) \qquad \left(\text{for } \begin{matrix} g, a, b \in G \\ x, y \in X \end{matrix} \right)$$

$\Leftrightarrow$ $ga = b$, $gax = by$

$\Leftrightarrow$ $ga = b$, $x = y$

hence $n \cdot m$ orbits in E^+.

$\Rightarrow$ $\ell = m \cdot n - n + 1 = n(m-1)+1$ $\square$

<u>Corollary</u> If G is a finitely generated group with m generators and if $H \subseteq G$ is a subgroup of index $[G : H] = n$, then H can be generated by $n(m-1)+1$ elements.

$\square$

This improves §1.13.

pf (There) is a surjective homomorphism $f: F_m \to G$, $f^{-1}(H)$ has index n in F_m. Now apply the proposition. $\square$

19. Proposition Let $\varphi : \Gamma' \to \Gamma$ be an immersion, the following are equivalent.

(i) $\varphi : |\Gamma'| \to |\Gamma|$ is a covering map.

(ii) for each vertex v in Γ', the restriction $st(\Gamma', v) \to st(\Gamma, \varphi(v))$ is bijective

(iii) For each vertex v in Γ', the restriction $U_{\frac{1}{2}}(v) \to U_{\frac{1}{2}}(\varphi(v))$ is bijective.

pf We have (i) => (iii) <=> (ii) ((iii => (i)): If $p \in |\Gamma|$ is the midpoint of some edge, $p = e_{\frac{1}{2}}$, then $\varphi^{-1}(U_{\frac{1}{2}}(p))$ is a disjoint union of open subsets $U_{\frac{1}{2}}(x) \subseteq a$, $x = a_{\frac{1}{2}}$, a edge, hence (iii) => (i). $\square$

20. <u>Theorem</u> Let $N \trianglelefteq F_m$ be a normal subgrp, $N \neq \{e\}$. The following are equivalent.

(i) $[F_m : N] < \infty$, N has finite index

(ii) N is finitely generated.

<u>pf</u> (i) $\Rightarrow$ (ii) by §7.18 or §1.13.

(ii) $\Rightarrow$ (i). Let R be a rose with m petals. We put $F_m = \pi_1(|R|;p)$. By §7.16 there is a finite connected graph Δ and an immersion $\varphi : \Delta \to R$ with $\varphi_* \pi_1(|\Delta|;x) = N$.

<u>We claim that $\varphi : |\Delta| \to |R|$ is a covering map.</u>

If this is proven, then $[F_m : N] = \#\varphi^{-1}(p)$ is finite by §6.12.

proof of the claim. (a) If y is any vertex in Δ, then $\varphi_* \pi_1(|\Delta|;y) = N$.

Let $\alpha \in \Omega(|\Delta|;x,y)$, consider.

$$\begin{tikzcd}
\pi_1(|\Delta|;x) \arrow[r, "\varphi_*"] \arrow[d, "C_{[\alpha]}"'] & \pi_1(|R|;p) \arrow[d, "C_{[\varphi\circ\alpha]}"] \\
\pi_1(|\Delta|;y) \arrow[r, "\varphi_*"] & \pi_1(|R|;p)
\end{tikzcd}$$

$C_{[\varphi\circ\alpha]} \leftarrow$ conjugation by $[\varphi\circ\alpha]$

Since $N \trianglelefteq F_m$, $C_{[\varphi\circ\alpha]}(N) = N$.

(b) Now we show that φ is surjective on stars.
Let v be any vertex in Δ, let $[\gamma] \in N$ be a non-trivial reduced edge path, let a denote the first edge in γ. Consider the reduced edge path $\tilde{\gamma} \in \Omega(|\Delta|; v)$ with $\varphi_*[\tilde{\gamma}] = [\gamma]$. Then $\varphi\circ\tilde{\gamma} = \gamma$ (uniqueness of reduced edge path, §7.8 Cor. C). Hence if $\tilde{a}$ is the first edge in $\tilde{\gamma}$, then $\varphi(\tilde{a}) = a$.
Let now b be any edge in R. If b or $\bar{b}$ the first (or last) edge in γ, then b has a preimage in Δ. Otherwise consider the reduced edge path $[\gamma_b] * [\gamma] * [\gamma_{\bar{b}}]$ which starts with b so b has a preimage.

Hence φ is a covering map.

<u>Corollary A</u> If $m \geq 2$, then the commutator subgroup $DF_m \trianglelefteq F_m$ is <u>not</u> finitely generated.

<u>pf</u> $F_m / DF_m \cong \mathbb{Z}^m$ by §1.11, and $DF_m \neq \{e\}$ since F_m is not abelian ($m \geq 2$!)

<u>Corollary B</u> Let $n \geq m \geq 2$. Then there is an injective homomorphism $F_n \to F_m$.

<u>pf</u> Let $X \subseteq F_m$ be a basis for DF_m. Then X is infinite, let $Y \subseteq X$ with $\# Y = n$. Then $F(Y) \cong \langle Y \rangle \subseteq F_m$ □

21. <u>Def</u> Let $\Gamma_1 \xrightarrow{\xi_1} \Gamma \xleftarrow{\xi_2} \Gamma_2$ be morphisms of graphs, $\Gamma_i = (V_i, E_i)$. We define a graph $\hat{\Gamma} = (\hat{V}, \hat{E})$, $\hat{V} = \{ (v_1, v_2) \in V_1 \times V_2 \mid \xi_1(v_1) = \xi_2(v_2) \}$

$$\hat{E} = \{ (a_1, a_2) \in E_1 \times E_2 \mid \xi_1(a_1) = \xi_2(a_2) \}$$

with morphisms $\varphi_i(x_1, x_2) = x_i$

$$\begin{tikzcd} \hat{\Gamma} \arrow[r, "\varphi_2"] \arrow[d, "\varphi_1"'] & \Gamma_2 \arrow[d, "\xi_2"] \\ \Gamma_1 \arrow[r, "\xi_1"'] & \Gamma \end{tikzcd}$$

(This is a pull-back in the category of graphs.)

<u>Lemma</u> If ξ_1, ξ_2 as above are immersions, let $(p_1, p_2) \in \hat{V}$. Then φ_1, φ_2 are immersions

and

$$(\xi_1 \circ \varphi_1)_* \pi_1(|\hat{\Gamma}|, (p_1,p_2)) \overset{①}{=} (\xi_1)_* \pi_1(|\Gamma_1|, p_1) \cap (\xi_2)_* \pi_1(|\Gamma_2|, p_2)$$

$$= (\xi_2 \circ \varphi_2)_* \pi_1(|\hat{\Gamma}|, (p_1,p_2)).$$

<u>pf</u> Clearly we have "$\subseteq$" at ①

Suppose $[\gamma]$ is in the right-hand side at ①.

We choose reduced edge paths γ_i in Γ_i (from p_i to p_i) with $(\xi_i)_* [\gamma_i] = [\gamma]$. But then

$\hat{\gamma} = (\gamma_1, \gamma_2)$ is a reduced edge path in $\hat{\Gamma}$, and

$(\xi_1 \circ \varphi_1)_* [\hat{\gamma}] = [\gamma]$. Hence "$\supseteq$" holds at ①.

<u>Claim: φ_1 is an immersion.</u>

If there are edges $a_i, b_i \in \Gamma_i$ such that

$(a_1, a_2), (b_1, b_2) \in \hat{E}$, $\quad (a_1)_0 = (b_1)_0$
$\qquad\qquad\qquad\qquad\qquad (a_2)_0 = (b_2)_0$

$\varphi_1(a_1, a_2) = \varphi_1(b_1, b_2)$, then $a_1 = b_1$

Now $\xi_2(a_2) = \xi_1(a_1) = \xi_1(b_1) = \xi_2(b_2)$

and ξ_2 is an immersion, hence $a_2 = b_2$. $\square$

22. <u>Theorem</u> <u>(Howson's Theorem)</u>

Suppose that $K_1, K_2 \leq F_m$ are finitely generated subgps. Then $K_1 \cap K_2$ is also finitely generated.

<u>Pf</u> We construct immersions $\Delta_i \xrightarrow{\xi_i} R$,

where R is a rose with m petals,

$F_m \cong \pi_1(|R|;p)$, $\pi_1(|\Delta_i|;p_i) \xrightarrow{\cong} K_i \leq F_m$

Then $\hat{\Delta}$ as in §7.21 is a <u>finite</u> graph,

hence $\pi_1(|\tilde{\Delta}|;(p_1,p_2)) \cong K_1 \cap K_2$ is a finitely generated free group. $\square$

<u>Remark</u> This result is <u>not</u> true for subgrps of finitely generated grps in general.

Put $A = F(\{a_1, a_2\}) \cong F_2$ $\quad B = F(\{b_1, b_2\}) \cong F_2$

Fix an isomorphism $DA \cong DB =: G$

and consider $G = A *_C B$. Then

$A, B \hookrightarrow G$ can be viewed as subgroups of G

and $A \cap B \cong C$ is not finitely generated. $\square$