§6. Fundamental groups

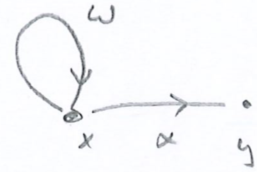
1. **Def** A <u>groupoid</u> is a small category Π, where every arrow is an isomorphism.

Ex Every group G "is" a groupoid, with one object $\{*\}$ and the group elements as morphisms. 

$g \circ h = gh$

In a groupoid Π, we have for each $x \in \mathrm{obj}(\Pi)$ the group $\Pi(x) = \mathrm{mor}_\Pi(x,x)$. If $x \xrightarrow{\alpha} y$ is an arrow, then $\Pi(x) \cong \Pi(y)$ via

$$c_\alpha : \omega \longmapsto \alpha \circ \omega \circ \alpha^{-1}$$



with inverse $(c_\alpha)^{-1} = c_{(\alpha^{-1})}$

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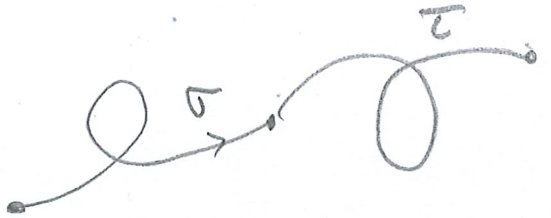
2. **Def** Let X be a topological space (eg. a metric space). A <u>path</u> σ from p to q in X is a continuous map

$$\sigma : [0,1] \to X \quad \text{with } \sigma(0) = p, \ \sigma(1) = q.$$

Suppose that $\tau : [0,1] \to X$ is another path in X, with $\tau(0) = \sigma(1)$. Then we define

$$\tau * \sigma(t) = \begin{cases} \sigma(2t) & 0 \le t \le \frac{1}{2} \\ \tau(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

path from $\sigma(0)$ to $\tau(1)$



We put $\Omega(X;p,q) = \{\alpha : [0,1] \to X \mid \alpha \text{ continuous}, \begin{array}{l}\alpha(0)=p\\ \alpha(1)=q\end{array}\}$

The composition $* : \Omega(X;q,r) \times \Omega(X;p,q) \to \Omega(X;p,r)$ is <u>not</u> associative. We fix this as follows.

3. Let X, Y be topological spaces, let $A \subseteq X$. Two (continuous) maps $\alpha, \beta : X \rightrightarrows Y$ are <u>homotopic rel A</u> if there is a continuous map $h : X \times [0,1] \to Y$

$$(x,s) \mapsto h_s(x)$$

with $h_0 = \alpha$, $h_1 = \beta$ <u>and</u> $h_t(a) = h_0(a)$ for all $a \in A$ (the latter is void if $A = \emptyset$). Homotopy rel A is an equivalence relation, we write <u>$\alpha \simeq \beta$ rel A</u>.

For the intervall $[0,1]$ we put $\partial = \{0,1\} \subseteq [0,1]$, and for $\alpha \in \Omega(X;p,q)$ we put

$$[\alpha] = \{ \beta \in \Omega(X;p,q) \mid \alpha \simeq \beta \text{ rel } \partial \}$$

($h_s(0) = h_0(0)$ and $h_s(1) = h_0(1)$ for all s).

Furthermore we put $\bar{\alpha}(t) = \alpha(1-t)$ $\varepsilon_p(t) = p$ $t \in [0,1]$

4. Proposition Let X be a topological space. Then we have the following.

(i) $\left.\begin{array}{ll} \alpha \simeq \alpha' \text{ rel } \partial & \text{in } \Omega(X; p, q) \\ \beta \simeq \beta' \text{ rel } \partial & \text{in } \Omega(X; q, r) \end{array}\right\} \Rightarrow \beta * \alpha \simeq \beta' * \alpha' \text{ rel } \partial$

hence we have a well-defined composition

$$[\beta] * [\alpha] = [\beta * \alpha]$$

(ii) $[\alpha * \varepsilon_p] = [\alpha]$ when $\alpha(0) = p$

$[\varepsilon_q * \alpha] = [\alpha]$ $\alpha(1) = q$

(iii) $[\bar{\alpha} * \alpha] = [\varepsilon_p]$

$[\alpha * \bar{\alpha}] = [\varepsilon_q]$

(iv) $(\gamma * \beta) * \alpha \simeq \gamma * (\beta * \alpha) \text{ rel } \partial$,

whence $([\gamma] * [\beta]) * [\alpha] = [\gamma] * ([\beta] * [\alpha])$

For a proof, see Spanier § I.7 or Hatcher 1.1. The proofs are explicit and not difficult.

5. **Def** The <u>fundamental groupoid</u> of a topological space X has the points of X as objects, and for p, q the set of all homotopy classes of paths $[\alpha]$ with $\alpha(0) = p$, $\alpha(1) = q$ as morphisms from p to q. (!).

We put $\pi_1(X; p, q) = \{ [\alpha] \mid \alpha \in \Omega(X; p, q) \}$

and $\pi_1(X; p) = \pi_1(X; p, p)$ for short.

This is the <u>fundamental group</u> of X with respect to the base point p.

The identity map at q is $[\varepsilon_q]$, the inverse of $[\alpha]$ is $[\bar{\alpha}]$. If σ is a path from p to q, then we obtain an isomorphism

$$c_{[\sigma]} : \pi_1(X; p) \xrightarrow{\cong} \pi_1(X; q)$$
$$[\alpha] \longmapsto [\sigma] * [\alpha] * [\bar{\sigma}]$$

⌜ Our convention for concatenation of paths follows <u>differs</u> from Spanier's, who does concatenation from left to right. $[\beta] \circ [\alpha]$

product in groupoid

! ⌟

If $f : X \to Y$ is a continuous map, we obtain a functor f_* between the fundamental groupoids via
$$f_*(p) = f(p)$$
$$f_*[\alpha] = [f \circ \alpha]$$

and in particular a group homomorphism

$$f_* : \pi_1(X; p) \to \pi_1(Y; f(p))$$

for each $p \in X$.

6. <u>Proposition</u> Let $(X_j)_{j \in I}$ be a family of topological spaces, with points $p_j \in X_j$, $j \in I$.
Put $P = \prod_{i \in I} X_i$, $p = (p_i)_{i \in I} \in P$.
Then the map

$$\pi_1(P; p) \longrightarrow \prod_{i \in I} \pi_1(X_i, p_i)$$

$$[\alpha] \longmapsto ([pr_i \circ \alpha])_{i \in I}$$

is a group isomorphism.

pf The map is by definition a group homomorphism, since $[pr_i \circ \alpha] = (pr_i)_* [\alpha]$ and the $pr_i : P \to X_i$ are continuous, with $pr_i(p) = p_i$.

Suppose $[\alpha]$ is in the kernel. Hence we have for each i a homotopy $h_i : [0,1] \times [0,1] \to X_i$ rel ∂

$h_{i,0}(t) = pr_i \alpha(t) = \alpha_i(t)$

$h_{i,1}(t) = p_i$

Put $h : [0,1] \times [0,1] \to P$

$(t,s) \longmapsto (h_{i,s}(t))_{i \in I} = h_s(t)$

$h_0(t) = \alpha(t)$, $h_1(t) = p \Rightarrow [\alpha] = [\varepsilon_p]$.

Hence the homomorphism is injective.

Suppose that $([\alpha_i])_{i \in I}$ is in the product.

Define $\alpha(t) = (\alpha_i(t))_{i \in I} \Rightarrow \alpha_i = pr_i \circ \alpha$

when $[\alpha_i] = [pr_i \circ \alpha] = (pr_i)_* [\alpha] \Rightarrow$ the homomorphism is surjective.

$\square$

7. <u>Def</u> A continuous map $f: X \to Y$ is called a <u>homotopy equivalence</u> if there is a continuous map $g: Y \to X$ such that

$$f \circ g \simeq id_Y \text{ rel } \emptyset$$

$$g \circ f \simeq id_X \text{ rel } \emptyset$$

Then X and Y are called <u>homotopy equivalent</u>, and g is called a <u>homotopy inverse</u> for f.

A topological space X is called <u>contractible</u> if id_X is homotopic (rel $\emptyset$) to a <u>constant</u> map $X \to \{p\} \subseteq X$.

<u>Lemma</u> A topological space X is contractible if and only if it is homotopy equivalent to a one-point space.

<u>pf</u> Suppose X is contractible. Hence there is a continuous map $h: X \times [0,1] \to X$, $h_0 = id_X$, $h_1(X) = \{p\} \subseteq X$.

Consider $i: \{p\} \subseteq X$ and $h_1: X \to \{p\}$, then

$$h_1 \circ i = id_{\{p\}} \quad \text{and} \quad i \circ h_1 = h_1 \simeq h_0 = id_X$$

$\Rightarrow$ X and $\{p\}$ are homotopy equivalent.

Suppose $Y = \{q\}$ and $f: X \to Y$ is a homotopy equivalence, with homotopy inverse

$g: Y \to X$, $g(q) = p \in X$. Then $g \circ f(x) = p$ for all $x \in X$ and $g \circ f \simeq id_X$ rel ϕ $\square$

8. <u>Proposition</u> Suppose that $f: X \to Y$ is a homotopy equivalence and that $p \in X$. Then $f_*: \pi_1(X;p) \to \pi_1(Y; f(p))$ is an isomorphism. #

<u>pf</u> The proof would be simpler if we could assume that the homotopies fix p and $f(p)$. We first prove an auxiliary result.

<u>Lemma</u> Let $h: X \times [0,1] \to Y$ be a continuous map, let $p \in X$ and put $\gamma(t) = h_t(p)$. Then the diagram

$$\begin{tikzcd}
 & \pi_1(Y; h_1(p)) \arrow[dd, "c_{[\bar\gamma]}", "\cong"'] \\
\pi_1(X;p) \arrow[ur, "(h_1)_*"] \arrow[dr, "(h_0)_*"'] & \\
 & \pi_1(Y; h_0(p))
\end{tikzcd}$$

commutes, $c_{[\bar\gamma]}([\beta]) = [\bar\gamma] * [\beta] * [\gamma]$.

pf Put $\gamma_s(t) = h_{st}(p) \Rightarrow \gamma_1 = \gamma$ and $\gamma_0 = \varepsilon_{h_0(p)}$. For $\alpha \in \Omega(X;p)$, the map

$$H_s(t) = ((\bar{\gamma}_s * (h_s \circ \alpha)) * \gamma_s)(t) \quad \text{is continuous (!)}$$

$\gamma_s(1) = h_s(p)$ $\quad (h_s \circ \alpha)(0) = h_s(p) = (h_s \circ \alpha)(1)$

$\bar{\gamma}_s(0) = h_s(p)$ $\quad$ and $H_s(0) = h_0(p) = H_s(1)$ for all s

$H_0(t) = (((\varepsilon_{h_0(p)} * (h_0 \circ \alpha)) * \varepsilon_{h_0(p)}))(t) \simeq h_0 \circ \alpha$ rel ∂

$H_1(t) = (((\bar{\gamma}_1 * (h_1 \circ \alpha)) * \gamma)(t)$

whence $(h_0)_*[\alpha] = [\bar{\gamma}] * ((h_1)_*[\alpha]) * [\gamma]$ $\square$

pf of the proposition: Let $g: Y \to X$ be a homotopy inverse to f and consider the homomorphisms

$$\pi_1(X;p) \xrightarrow[\text{1}]{f_*} \pi_1(Y;f(p)) \xrightarrow[\text{2}]{g_*} \pi_1(X;gf(p)) \xrightarrow[\text{3}]{f_*} \pi_1(Y;fgf(p))$$

ξ $\quad\quad$ η

By the lemma, ξ is an isomorphism and η is an isomorphism. Thus 1 is injective, 2 is injective, 2 is surjective, 2 is isomorphism

$\Rightarrow$ 1 is an isomorphism. $\square$

<u>Corollary</u> If X is contractible, then $\pi_1(X;p) = \{[\varepsilon_p]\}$ is the trivial group, for every $p \in X$. $\square$

Now we consider covering spaces.

9. <u>Def</u> A continuous map $E \xrightarrow{f} X$ is called a covering map if every $p \in X$ has an open neighborhood $U \subseteq X$ which is <u>evenly covered by f</u>, that is:

$f^{-1}(U) = \bigcup_{i \in I} U_i$ $\quad$ I some index set (depending possibly on p)

U_i <u>open</u> in E, $U_i \cap U_j = \emptyset$ for $i \neq j$,

$U_i \xrightarrow{f} U$ a homeomorphism for each i

<u>Examples</u> (1) $X = E$, $f = id_X$

(2) D some discrete set, $E = X \times D$,

$f(x, d) = x$

(3) $X = \{ z \in \mathbb{C} \mid |z| = 1 \} = S^1 \subseteq \mathbb{C}$

$E = \mathbb{R}$, $f(t) = \exp(2\pi i t)$ $\quad i = \sqrt{-1}$

f is a continuous group homomorphism with kernel $\mathbb{Z} \subseteq \mathbb{R}$ $\quad (f(t) = 1 \Leftrightarrow t \in \mathbb{Z})$

$f^{-1}(1) = \mathbb{Z}$, $\quad f^{-1}(-1) = \frac{1}{2} + \mathbb{Z}$

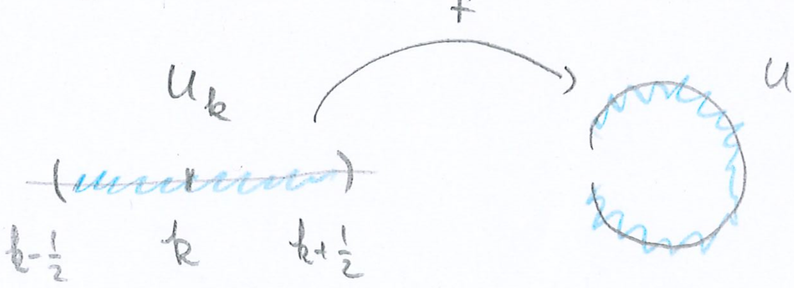
<u>Claim</u> This is a covering map.

(a) $p \neq -1 \rightsquigarrow$ put $U = S^1 - \{-1\}$

$f^{-1}(U) = \mathbb{R} - (\frac{1}{2} + \mathbb{Z})$

for each interval $U_k = (k - \frac{1}{2}, k + \frac{1}{2})$ $\quad k \in \mathbb{Z}$

$U_k \longrightarrow U$ is continuous, bijective



with <u>derivative $f' \neq 0$</u> $\Rightarrow$ f is open map

$\Rightarrow$ f is homeomorphism $U_k \xrightarrow{\cong} U$

Similarly for $p = -1$, $U = S^1 - \{1\}$

$U_k = (k, k+1) \subseteq \mathbb{R}$

<u>Def</u> let

$$\begin{tikzcd} & E \arrow[d, "f"] \\ Z \arrow[r, "h"] & X \end{tikzcd}$$

be continuous maps.

A <u>lift</u> of h is a continuous map $\tilde{h}: Z \to E$ with $f \circ \tilde{h} = h$,

$$\begin{tikzcd} & E \arrow[d, "f"] \\ Z \arrow[ur, "\tilde{h}"] \arrow[r, "h"] & X \end{tikzcd}$$

10. <u>Theorem</u> let $E \xrightarrow{f} X$ be a covering map. Suppose that $Z \times [0,1] \xrightarrow{h} X$ is a continuous map and that $h_0 : Z \to X$ admits a lift $\tilde{h}_0 : Z \to E$. Then h has a <u>unique</u> lift $\tilde{h} : Z \times [0,1] \to E$ extending $\tilde{h}_0$.

The proof will be given below. We first derive consequences of the theorem.

Throughout, $E \xrightarrow{f} X$ is assumed to be a covering map.

<u>Corollary A</u> Suppose $\alpha : [0,1] \to X$ is a path, and that $q \in E$ with $f(q) = \alpha(0)$. Then α has a unique lift $\tilde{\alpha} : [0,1] \to E$ with $\tilde{\alpha}(0) = q$.

<u>Corollary B</u> Suppose that $h : [0,1] \times [0,1] \to X$ is continuous and that $q \in E$ with $f(q) = h_0(0)$. Then h has a unique lift $\tilde{h}$ with $\tilde{h}_0(0) = q$.

<u>pf</u> Use Cor A to lift h_0 to $\tilde{h}_0$ and then the theorem to lift h to $\tilde{h}$. $\square$

<u>Corollary C</u> Suppose that $q \in E$. Then $f_* : \pi_1(E; q) \to \pi_1(X; f(q))$ is an injective homomorphism.

pf Suppose $\alpha \in \Omega(E; q, q)$ with $f_*[\alpha] = [\varepsilon_p]$
$p = f(q)$. Hence there is $h : [0,1] \times [0,1] \to X$ cts.
with $h_0 = f \circ \alpha$, $h_1 = \varepsilon_p$. Let $\tilde{h}$ denote
its unique lift, with $\tilde{h}_0(0) = q$. Then
$\tilde{h}_0 = \alpha$ (by uniqueness), $\tilde{h}_1 = \varepsilon_q$ (by uniqueness)
$\tilde{h}_s(0) = q = \tilde{h}_s(1)$ (by uniqueness)
$\Rightarrow [\alpha] = [\varepsilon_q]$ $\square$

Corollary D For $\mathbb{S}^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$
there is an isomorphism

$$\mathbb{Z} \longrightarrow \pi_1(\mathbb{S}^1; 1)$$
$$k \longmapsto [\alpha_k]$$

where $\alpha_k(t) = \exp(2\pi i t k)$ $i = \sqrt{-1}$

pf We have $\alpha_k \in \Omega(\mathbb{S}^1; 1, 1)$. We consider
the covering $\mathbb{R} \xrightarrow{f} \mathbb{S}^1$, $t \mapsto \exp(2\pi i t)$ $i = \sqrt{-1}$
Then $\tilde{\alpha}_k(t) = kt$ is the unique lift of α_k
with $\tilde{\alpha}_k(0) = 0$.
Let $[\beta] \in \pi_1(\mathbb{S}^1; 1)$, let $\tilde{\beta}$ denote the unique
lift of β with $\tilde{\beta}(0) = 0$.

Then $\tilde{\beta}(1) = k$ for some $k \in \mathbb{Z}$, because $f(\tilde{\beta}(1)) = 1$. Then $\tilde{\alpha}_k \simeq \beta$ rel ∂ via

$\tilde{h}_s(t) = s\,\tilde{\alpha}_k(t) + (1-s)\,\beta(t)$, whence $[\beta] = [\alpha_k]$ (via $h_s(t) = f(\tilde{h}_s(t))$). This shows that the map $k \mapsto [\alpha_k]$ is a bijection

$$\mathbb{Z} \to \pi_1(S^1; 1)$$

(a bijection because $\tilde{\alpha}_k(1) \neq \tilde{\alpha}_l(1)$ for $k \neq l$)

If we put $\tilde{\alpha}_{l,k}(t) = k + tl$, then $\tilde{\alpha}_{l,0} = \tilde{\alpha}_l$ and

$\tilde{\alpha}_{l,k} * \tilde{\alpha}_{k,0}$ is a lift of $\alpha_l * \alpha_k$, whence

$$[\alpha_l] * [\alpha_k] = [\alpha_{k+l}]$$

$\square$

Now we prove the theorem. We first show that Cor. A holds. Let $\alpha : [0,1] \to X$ be a path in X. Since $[0,1]$ is compact, we find $s_0 = 0 < s_1 < s_2 < \ldots < s_m = 1$ such that $\alpha([s_{k-1}, s_k]) \subseteq U_k$ and U_k is evenly covered,

$$\begin{array}{ccc} & & V_k \subseteq E \\ & & \text{homeo } \cong \big\downarrow f \\ [s_{k-1}, s_k] & \xrightarrow{\alpha} & U_k \subseteq X \end{array}$$

We define $\tilde{\alpha}$ inductively on $[s_{k-1}, s_k]$ as follows.

<u>$k=1$</u> Choose $V_1 \subseteq f^{-1}(U_1)$ so that $q \in V_1$ and use the homeomorphism $U_1 \xrightarrow[\cong]{\varphi_1} V_1$, $\tilde{\alpha}(t) = \varphi_1 \circ \alpha$, for $0 \leq t \leq s_1$

<u>$k \geq 2$</u> Choose $V_k \subseteq f^{-1}(U_k)$ so that $\tilde{\alpha}(s_{k-1}) \in V_k$ and use the homeomorphism $U_k \xrightarrow[\cong]{\varphi_k} V_k$, $\tilde{\alpha}(t) = \varphi_k \circ (t)$ for $s_{k-1} \leq t \leq s_k$

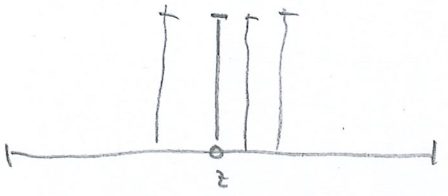
This shows the <u>existence</u> of the lift.

Suppose that $\beta : [0,1] \to E$ is a lift of α, with $\beta(0) = q$. Since $\beta(0) \in V_1$ and $\beta([0,s_1])$ is connected, $\beta([0,s_1]) \subseteq V_1$ (because $f^{-1}(U_1)$ is disjoint union of open sets). From the bijection $V_1 \xrightarrow{f} U_1$ we obtain that $\beta(t) = \tilde{\alpha}(t)$ for $0 \leq t \leq s_1$. Now we continue like this, $\beta([s_{k-1}, s_k]) \subseteq V_k$ (by connectedness of $[s_{k-1}, s_k]$) $\Rightarrow \beta(t) = \tilde{\alpha}(t)$ for $s_{k-1} \leq t \leq s_k$ $\square$

Now for the general case, as stated in the Theorem.

For each $z \in Z$, we extend $\tilde{h}_0$ to $\{z\} \times [0,1]$ (by the result proved so far). Thus we obtain a map $\tilde{h} : Z \times [0,1] \to E$ with $\tilde{h}_0 = \tilde{h}_0$

We have to show this map is <u>continuous</u>.



$$\begin{tikzcd} & E \arrow[d] \\ Z \arrow[ur, bend left, "\tilde{h}"] \arrow[r, bend right, "h"] & X \end{tikzcd}$$

Let $(z, s) \in Z \times [0,1]$. <u>Claim</u>: $\tilde{h}$ is continuous at (z,s).

Since $\{z\} \times [0,1] \subseteq Z \times [0,1]$ is compact, we find an open neighborhood $V \subseteq Z$ of z and $s_0 = 0 < s_1 < \dots < s_m = 1$ such that $h(V \times [s_{k-1}, s_k]) \subseteq X$ is in some evenly covered open set. Then $\tilde{h}$ is continuous on $V \times [0, s_1]$, $V \times [s_1, s_2]$, $\dots$, $V \times [s_{m-1}, s_m]$ by its construction and hence continuous on $V \times [0,1]$.

Hence $\tilde{h}$ is continuous.

If $h' : Z \times [0,1] \to E$ is any lift of h, extending $\tilde{h}_0 : Z \to E$, then $h'|_{\{z\} \times [0,1]} = \tilde{h}|_{\{z\} \times [0,1]}$ by Cor A and hence $h' = \tilde{h}$

$\square$

11. Some Observations

(1) A topological space X is path connected if for all $p, q \in X$ there is a path $\alpha : [0,1] \to X$ with $\alpha(0) = p$ and $\alpha(1) = q$. If $f : X \to Y$ is continuous and if X is path connected, then $f(X) \subseteq Y$ is path connected.

(2) Every covering map $E \xrightarrow{f} X$ is an open map (images of open sets are open).

(3) If $E \xrightarrow{f} X$ is a covering map and if X is path connected, then $\# f^{-1}(p) = \# f^{-1}(q)$ for all $p, q \in X$.

12. Proposition

Let $E \xrightarrow{f} X$ be a covering map. If E is path connected, then we have for $q \in E$ and $p = f(q)$

$$\left[\pi_1(X;p) : f_* \pi_1(E;q)\right] = \# f^{-1}(p)$$

Note: the left hand side is independent of $q \in f^{-1}(p)$.

Pf For $\alpha \in \Omega(X;p)$ let $\tilde{\alpha}$ denote the unique lift with $\tilde{\alpha}(0) = q$. We put

$$\mathscr{X} : \pi_1(X;p) \longrightarrow f^{-1}(p),$$
$$[\alpha] \longmapsto \tilde{\alpha}(1)$$

Then $\varkappa$ is surjective, since E is path connected. We have

$$\varkappa([\alpha]) = \varkappa([\beta]) \iff \tilde{\alpha}(1) = \tilde{\beta}(1)$$

If $\tilde{\alpha}(1) = \tilde{\beta}(1)$, then $\tilde{\bar{\alpha}} * \tilde{\beta} = \widetilde{\bar{\alpha} * \beta}$, hence

$[\bar{\alpha}] * [\beta] \in f_* \pi_1(E;q) \Rightarrow [\beta] = [\alpha] * [\gamma]$ for some $[\gamma] \in \pi_1(E;q)$.

Conversely, if $[\alpha] = [\beta] * [\gamma]$ for $[\gamma] \in \pi_1(E;q)$, then $\tilde{\alpha} \simeq \widetilde{\beta * \gamma}$ and $\tilde{\beta} * \tilde{\gamma} = \widetilde{\beta * \gamma}$, hence $\tilde{\alpha}(1) = \tilde{\beta}(1)$. Hence

$$\varkappa([\alpha]) = \varkappa([\beta]) \iff [\alpha] * f_* \pi_1(E;q) = [\beta] * f_* \pi_1(E;q)$$

and we have a bijection $f^{-1}(p) \to \pi_1(X;p) / f_* \pi_1(E;q)$ $\square$

13. Def A topological space Z is locally path connected if for every $z \in Z$ and every neighborhood $W \subseteq Z$ of z there is an open path connected set U, with $z \in U \subseteq W$

For example, every open set $Z \subseteq \mathbb{R}^m$ is locally path connected.

Theorem Let $E \xrightarrow{f} X$ be a covering map, let $h : Z \to X$ be a continuous map, with $z_0 \in Z$, $q \in E$ such that $h(z_0) = f(q)$. If Z is path connected and locally path connected then the following are equivalent.

(i) h admits a lift $\tilde{h} : Z \to E$, with $\tilde{h}(z_0) = q$

(ii) $h_* \pi_1(Z; z_0) \subseteq f_* \pi_1(E; q)$.

If these hold, then $\tilde{h}$ is unique.

Pf Suppose that a lift $\tilde{h}$ exists. Then $h_* = f_* \tilde{h}_*$ and (ii) follows.

Given $z \in Z$, there is some $\xi \in \Omega(Z; z_0, z)$. Then $h \circ \xi$ has a unique lift $\widetilde{h \circ \xi}$ with $\widetilde{h \circ \xi}(0) = q$ and hence $\tilde{h}(z) = \widetilde{h \circ \xi}(1)$, because $\tilde{h} \circ \xi$ is a lift of $h \circ \xi$. Hence $\tilde{h}$ is unique.

Suppose that (ii) holds. Let $z \in Z$ and choose $\xi \in \Omega(Z; z_0, z)$. Put $u = \widetilde{h \circ \xi}(1)$ where $\widetilde{h \circ \xi}$ is a lift of $h \circ \xi$ starting at q. Note that u depends only on the homotopy class of ξ rel ∂.

If $\bar{\xi} \in \Omega(Z; z_0, z)$ is another path, then

$\bar{\xi} * \xi \in \Omega(Z; z_0)$ and $h_*[\bar{\xi} * \xi] \in f_* \pi_1(E; q)$.

Hence $h \circ (\bar{\xi} * \xi)$ lifts to an element in

$\Omega(E; q)$, whence $\widetilde{h \circ \bar{\xi}}(1) = \widetilde{h \circ \xi}(1)$.

This shows that we have a well-defined map

$\hat{h} : Z \to E$, putting $\hat{h}(z) = u$.

It remains to show that $\hat{h}$ is <u>continuous</u>.

Let $z \in Z$. Then z has an open neighborhood W which is path connected, such that $h(W) \subseteq U$ and $U \subseteq X$ is open and evenly covered.

For $w \in W$ there is a path $\alpha \in \Omega(W; z, w)$.

Let $\xi \in \Omega(Z; z_0, z)$. Let $V \subseteq F^{-1}(U)$ open, with $F : V \xrightarrow{\cong} U$ (U is evenly covered) and with $\hat{h}(z) \in V$. Put $g : U \xrightarrow{\cong} V$ and

consider $(g \circ h \circ \alpha) * \widetilde{h \circ \xi}$. This is

a lift of $(h \circ \alpha) * (h \circ \xi) = h \circ (\alpha * \xi)$,

hence $g(h(w)) = \hat{h}(w)$ and $\hat{h}$ is continuous

at z. $\square$

#

14. __Def__ Let $E \xrightarrow{f} X$ be a covering map.

A homeomorphism $g: E \to E$ is called a __deck transformation__ if $f = f \circ g$,

$$\begin{tikzcd} E \arrow[rr, "g"] \arrow[dr, "f"'] & & E \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

The deck transformations form a group

$Deck(E \xrightarrow{f} X) = \{ g: E \to E \mid g \text{ a deck transformation} \}$.

Recall also: If $H \leq G$ is a subgroup, then the __normalizer__ of H in G is the subgrp

$$Nor_G(H) = \{ n \in G \mid nHn^{-1} = H \}.$$

Thus $H \trianglelefteq Nor_G(H)$ and $Nor_G(H)$ is the largest subgrp of G that normalizes H.

15. __Theorem__ Let $E \xrightarrow{f} X$ be a covering map, where E is path connected and locally path connected.

Suppose that $f(u) = f(v) = p$ for $u, v \in E$.

The following are equivalent.

(i) There is $g \in Deck(E \xrightarrow{f} X)$ with $g(u) = v$

(iii) If $\alpha \in \Omega(E;u,v)$ then $[f\circ\alpha] \in N$,

where $N = N_{\pi_1(X;p)} f_* \pi_1(E;u)$.

If these conditions hold, then

$$\mathrm{Deck}(E \xrightarrow{f} X) \cong N/H$$

for $H = f_* \pi_1(E;u)$.

<u>pf (i) $\Rightarrow$ (ii)</u> Consider the diagrams

$$\begin{tikzcd} E \arrow[rr, "g", "\cong"'] \arrow[dr, "f"'] & & E \arrow[dl, "f"] \\ & X & \end{tikzcd} \qquad \begin{tikzcd} \pi_1(E;u) \arrow[rr, "g_*", "\cong"'] \arrow[dr, "f_*"'] & & \pi_1(E;v) \arrow[dl, "f_*"] \\ & \pi_1(X;p) & \end{tikzcd}$$

For $\alpha \in \Omega(E;u,v)$ and $\gamma \in \Omega(E,u)$ we have

$(\alpha * \gamma) * \bar{\alpha} \in \Omega(E;v)$ and

$$f_*[(\alpha*\gamma)*\bar{\alpha}] = [f\circ\alpha] * f_*[\gamma] * [f\circ\bar{\alpha}] \in f_*\pi_1(E;v)$$
$$= f_*\pi_1(E;u)$$

hence $[f\circ\alpha]$ normalizes $f_*\pi_1(E;u)$.

(ii) => (i) Suppose that $\alpha \in \Omega(E; u, v)$, with $[f\circ\alpha] \in N$ and consider the lifting problem

$$\begin{tikzcd} & E \arrow[d, "f"] \\ E \arrow[ur, dashed, "g"] \arrow[r, "f"] & X \end{tikzcd}$$

with $g(u) = v$.

Every $[\gamma] \in \pi_1(E; u)$ can be written as

$[\gamma] = [(\bar\alpha * \rho) * \alpha]$ for some $[\rho] \in \pi_1(E; v)$,

hence $f_* \pi_1(E; u) \subseteq f_* \pi_1(E; v)$, since

$$f_*[\gamma] = [f\circ\bar\alpha] * f_*[\rho] * [f\circ\alpha] \in f_*\pi_1(E; v).$$

By theorem §6.13, there is a lift g of f with $g(u) = v$. Similarly, there is a lift h of f with $h(v) = u$. Then $h\circ g$ solves

$$\begin{tikzcd} & E \arrow[d, "f"] \\ E \arrow[ur, "h\circ g"] \arrow[r, "f"'] & X \end{tikzcd}$$

$h\circ g(u) = u$

which is also solved by id_E, hence $h\circ g = id_E$

Similarly, $g\circ h = id_E$, hence g is a homeomorphism.

Now for the isomorphism.

For $[\alpha] \in N$ let $\tilde{\alpha}$ denote the unique lift with $\tilde{\alpha}(0) = u$. By the proof of (i) => (ii) there is a unique deck transformation g with

$$g(u) = \tilde{\alpha}(1), \quad \text{put} \quad \varphi([\alpha]) = g.$$

Suppose that $[\alpha], [\beta] \in N$. Then

$$\varphi([\alpha * \beta])(u) = \widetilde{\alpha * \beta}(1)$$

For $\tilde{\gamma} = \varphi([\beta]) \circ \tilde{\alpha}$ we have $\tilde{\gamma}(0) = \tilde{\beta}(1)$

$$\Rightarrow \quad \tilde{\gamma} * \tilde{\beta} = \widetilde{\alpha * \beta}$$

$$\varphi([\alpha * \beta])(u) = \tilde{\gamma}(1) = (\varphi([\beta]) \circ \tilde{\alpha})(1)$$

$$= \varphi([\beta]) \circ \varphi([\alpha])(u).$$

Thus $\varphi([\alpha * \beta]) = \varphi([\alpha] * [\beta]) = \varphi[\beta] * \varphi[\alpha]$.

Note: $\varphi[\alpha] = id \Leftrightarrow \tilde{\alpha}(1) = u \Leftrightarrow \alpha \in f_* \pi_1(E; u)$

The map φ is an <u>antihomomorphism</u>. There are two ways to fix this:

(a) consider $[\alpha] \longmapsto \varphi([\alpha])^{-1}$ or

(b) let $\mathrm{Deck}(E \xrightarrow{f} X)$ act from the <u>right</u> on E

In any case we obtain an injective homomorphism

$$N/H \longrightarrow \mathrm{Deck}(E \xrightarrow{f} X).$$

If $g \in \mathrm{Deck}(E \xrightarrow{f} X)$ with $g(u) = v$, then the diagram at the beginning of the proof shows for $\alpha \in \Omega(E; u, v)$ that $g = \varphi([f \circ \alpha])$, hence φ is surjective, and $N/H \cong \mathrm{Deck}(E \xrightarrow{f} X)$. $\square$

<u>Corollary A</u> (Assumptions as in Thm 15.) If $\pi_1(E;u) = \{[\varepsilon_u]\}$ is the trivial group, then $\mathrm{Deck}(E \xrightarrow{f} X) \cong \pi_1(X;p)$.

A covering map $E \xrightarrow{f} X$ is called <u>normal</u> if f is surjective and if $\mathrm{Deck}(E \xrightarrow{f} X)$ acts transitively on $f^{-1}(p)$, for all $p \in X$.

<u>Corollary B</u> (Assumptions as in Thm 15.) Assume that X is path connected. Then $f: E \to X$ is normal if and only if $f_* \pi_1(E;u) \trianglelefteq \pi_1(X, p)$.

⌜Note: it suffices that this holds for some $u \in E$, since X is path connected. ⌟

<u>Remark</u> If E is path connected, then $\mathrm{Deck}(E \xrightarrow{f} X)$ acts <u>freely</u> on $f^{-1}(p)$, for $p = f(u)$ and $u \in E$.

Suppose that g is a deck transformation fixing u.

Consider

$$\begin{tikzcd} & E \arrow[d, "f"] \\ E \arrow[ur, "g"] \arrow[r, "f"'] & X \end{tikzcd}$$

Given $w \in E$, let $\alpha \in \Omega(E; u, w)$. Then both α and $g \circ \alpha$ are lifts of $f \circ \alpha$ starting at u, hence $\alpha(1) = g \circ \alpha(1) \Rightarrow w = g(w)$ for all $w \in E$ $\square$

16. Def An action $G \times E \to E$ of a group G on a space E (by homeomorphisms) is called a <u>covering space action</u> if every $u \in E$ has an open neighbourhood U such that $U \cap g(U) = \emptyset$ for all $g \neq e$. Note: this implies that the action is <u>free</u>, no $g \neq e$ has a fixed point. In particular, the action is faithful.

17. <u>Theorem</u> Let $G \times E \to E$ be a covering space action. Put $X = {}_G\backslash E$ (the orbit space) with the quotient topology. Then the following hold.

(i) $E \xrightarrow{f} X$ is a normal covering map ($f(u) = G(u)$)

(ii) If E is path connected, then $\mathrm{Deck}(E \xrightarrow{f} G) \cong G$

(iii) If E is path connected and locally path connected, then $G \cong \pi_1(X;p) / f_* \pi_1(E;u)$

for $f(u) = p$.

pf For $p = f(u)$ let $U \subseteq E$ be an open neighborhood of u with $U \cap g(U) = \phi$ for all $g \neq e$. Put $W = f(U) \subseteq X$. Then W is open, with $f^{-1}(W) = \bigcup_{g \in G} g(U)$ and for each g,

$g(U) \to W$ is a homeomorphism; $g_1(U) \cap g_2(U) = \phi$ for $g_1 \neq g_2$.

Hence $f: E \to X$ is a covering map and each $g \in G$ acts as a deck transformation. Hence the covering map is normal. Also, $G \subseteq \mathrm{Deck}(E \xrightarrow{f} X)$.

If E is path connected, then $\mathrm{Deck}(E \xrightarrow{f} X)$ acts freely on $f^{-1}(p)$ by §6.15 (Remark p. $138\frac{1}{2}$) and thus $G = \mathrm{Deck}(E \xrightarrow{f} X)$.

The last claim is Thm §6.15.

$\square$

Corollary $\pi_1(S^1; 1) \cong \mathbb{Z}$.

pf The action $\mathbb{Z} \times \mathbb{R} \to \mathbb{R}$, $(k, s) \mapsto k+s$ is a covering space action. The map $f: \mathbb{R} \to S^1$, $t \mapsto \exp(2\pi i t)$ is open and hence a quotient map.

Hence ${}_{\mathbb{Z}}\backslash^{\mathbb{R}} \cong S^1$ and the result follows.

#

18. Def A topological space X is 1-connected if it is nonempty, path connected and if $\pi_1(X;p) = \{[c_p]\}$ is trivial for some (hence all) $p \in X$.

If $E \xrightarrow{f} X$ is a covering map and if E is locally path connected and 1-connected, then $\mathrm{Deck}(E \xrightarrow{f} X) \cong \pi_1(X;p)$ and $\mathrm{Deck}(E \xrightarrow{f} X)$ acts transitively and freely on $f^{-1}(p) \subseteq E$ by §6.14.

19. The Seifert - Van Kampen Theorem

We consider the following situation. X is a topological space, $p \in X$ and $\mathcal{U}$ is a collection of open subsets of X with the following properties.

(i) $X = \bigcup \mathcal{U}$ and $p \in \bigcap \mathcal{U}$

(ii) for all $U, V, W \in \mathcal{U}$, the open set $U \cap V \cap W$ is path connected. In particular, each $U \in \mathcal{U}$ is path connected.

We put $\rho_U : U \hookrightarrow X \qquad U \in \mathcal{U}$

$\rho_{U,V} : U \cap V \longrightarrow V \qquad U, V \in \mathcal{U}$

Lemma A The canonical homomorphism

$$\coprod_{U \in \mathcal{U}} \pi_1(U;p) \longrightarrow \pi_1(X;p) \quad \text{is surjective.}$$

pf Let $\alpha \in \Omega(X;p)$. Since $[0,1]$ is compact, there are $s_0 = 0 < s_1 < s_2 < \ldots < s_m = 1$ and $U_1, \ldots, U_m \in \mathcal{U}$ with $\alpha([s_{k-1}, s_k]) \subseteq U_k$, for $k = 1, \ldots, m$. We choose $\beta_k \in \Omega(U_k \cap U_{k+1}; p, \alpha(s_k))$

and put ~~$\alpha_k(t) =$~~

$\alpha_k(t) = \alpha(s_{k-1} + t(s_k - s_{k-1}))$

Then $[\alpha] = [\alpha_m * \beta_{m-1}] * [\bar{\beta}_{m-1} * (\alpha_{m-1} * \beta_{m-2})] * \ldots$

$* [\bar{\beta}_1 * \alpha_1]$

and $[\alpha_m * \beta_{m-1}] \in \pi_1(U_m; p)$

$$[\bar{\beta}_{m-1} * (\alpha_{m-1} * \beta_{m-2})] \in \pi_1(U_{m-1}; p)$$

$$\vdots$$

$$[\bar{\beta}_1 * \alpha_1] \in \pi_1(U_1; p) \qquad \square$$

<u>Corollary</u> Let $l \geq 2$ and put $S^l = \{v \in \mathbb{R}^{l+1} \mid \|v\|_2 = 1\}$.
Then $\pi_1(S^l; p) = \{[\varepsilon_p]\}$ is the trivial group.

<u>pf</u> Choose $q \neq \pm p$ in S^l, put

$$U_1 = S^l - \{q\} \qquad U_2 = S^l - \{-q\}$$

Then U_1, U_2 are contractible (homeomorphic to $\mathbb{R}^l$)
and $U_1 \cap U_2 = S^l - \{\pm q\}$ is path connected

$$\cong \mathbb{R} \times S^{l-1}$$

Hence $\coprod_{i=1,2} \pi_1(U_i; p) = \{e\}$ is the trivial gp. $\square$

Lemma B For $V \in \mathcal{U}$ consider the inclusion homomorphism

$$i_V : \pi_1(V;p) \longrightarrow \coprod_{U \in \mathcal{U}} \pi_1(U;p)$$

If $\alpha \in \Omega(U \cap V; p)$, $U, V \in \mathcal{U}$, then

$$i_U([\alpha]) \cdot i_V([\bar{\alpha}]) \in \ker \Phi.$$

pf We consider

$$\begin{tikzcd} & U \arrow[dr, "\rho_U"] & \\ U \cap V \arrow[ur, "\rho_{V,U}"] \arrow[dr, "\rho_{U,V}"'] & & X \\ & V \arrow[ur, "\rho_V"'] & \end{tikzcd}$$

$(\rho_U)_* (\rho_{V,U})_* [\alpha] = (\rho_V)_* (\rho_{V,U})_* [\alpha]$, whence

$$\Phi\big(i_U((\rho_{V,U})_*[\alpha])\big) = \Phi\big(i_V((\rho_{U,V})_*[\alpha])\big) \qquad \square$$

We put

$$L = \{ i_U[\alpha] \cdot i_V[\bar{\alpha}] \mid U, V \in \mathcal{U},\ \alpha \in \Omega(U,V;p) \}$$

<u>Theorem</u> (Seifert - Van Kampen)

The kernel of Φ is

$\ker \Phi = \langle\langle L \rangle\rangle$ and hence (by Lemma A)

$$\pi_1(X;p) \cong \coprod_{U \in \mathcal{U}} \pi_1(U;p) \Big/ \langle\langle L \rangle\rangle .$$

<u>pf</u> We know already that $L \subseteq \ker \Phi$. It remains to show that $\langle\langle L \rangle\rangle = \ker \Phi$.

By Lemma A, every element $[\alpha] \in \pi_1(X;p)$ can be written as

$$[\alpha] = \Phi\left(i_{U_n}[\alpha_n] \cdots i_{U_1}[\alpha_1] \right) \quad \text{for}$$

$$U_1, \dots, U_n \in \mathcal{U}, \quad \alpha_k \in \Omega(U_k;p), \quad n \geq 0$$

We call $i_{U_n}[\alpha_1] \cdots i_{U_1}[\alpha_1]$ a <u>factorization</u> of $[\alpha]$. Two factorizations are <u>equivalent</u> if they can be transformed into each other using the following operations.

(I) Replace $i_U[\alpha]\, i_U[\beta]$ by $i_U([\alpha] * [\beta])$

($\bar{I}$) The inverse of (I).

(II) If $\alpha \in \Omega(U \cap V; p)$, then replace

$i_U[\alpha]$ by $i_V[\alpha]$

<u>Note</u> (I) and ($\bar{I}$) do not change the element

$i_{U_m}[\alpha_m] \cdots i_{U_1}[\alpha_1]$ in the free product.

(II) does not change the <u>coset</u>

$i_{U_m}[\alpha_m] \cdots i_{U_1}[\alpha_1] \langle\langle L \rangle\rangle$ by the definition of L.

<u>Lemma C</u> If $\Phi(i_{U_m}[\alpha_m] \cdots i_{U_1}[\alpha_1]) =$

$\Phi(i_{V_n}[\beta_n] \cdots i_{V_1}[\beta_1])$, then these factorizations

are equivalent.

<u>Note</u> Lemma C shows that $\ker(\Phi) = \langle\langle L \rangle\rangle$

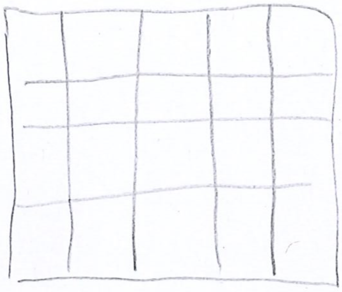
and finishes the proof of Seifert-Van Kampen.

<u>pf of Lemma C</u> By assumption, we have

a homotopy $h : [0,1] \times [0,1] \to X$

$\alpha_m * (\alpha_{m-1} * (\ldots * \alpha_1)) \simeq \beta_n * (\beta_{n-1} * (\ldots \beta_1)))$ rel ∂

From the compactness of $Q = [0,1] \times [0,1]$ we can cover Q by finitely many rectangles $A_1, \dots, A_N$ such that $h(A_k) \subseteq U_k \in \mathcal{U}$. Subdividing these rectangles further using all sides of the original rectangles, we obtain a decomposition of Q into "bricks".

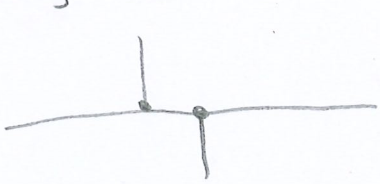


such that for each brick A, $h(A) \subseteq U_A \in \mathcal{U}$.

We may also assume that the points with x-coordinates $\frac{1}{2}, \frac{1}{4}, \dots, \frac{1}{2^p}$ $\quad p = \max\{m,n\}$

are on vertical lines (introduce more bricks).

Perturbing the vertical lines in the middle layers, we may assume that every vertex of a brick is in <u>at most 3 bricks</u>



and still $h(A) \subseteq U_A \in \mathcal{U}$.

For each vertex v contained in bricks A, B, C, we choose $\gamma_v \in \Omega(\mathcal{U}_A \cap \mathcal{U}_B \cap \mathcal{U}_C; p, h(v))$.
If we insert these paths, we obtain for each horizontal line in Q a factorization of $[h_0]$. For a path of the shape

in Q

we obtain also a factorization of $[h_0]$, which is equivalent to the factorization given by

The factorization in the top and bottom line can then be made equivalent to the factorization

$i_{V_n}[p_n]\, i_{V_{n-1}}[p_{n-1}] \cdots i_{V_1}[p_1]$ and

$i_{\mathcal{U}_m}[\alpha_m]\, i_{\mathcal{U}_{m-1}}[\alpha_{m-1}] \cdots i_{\mathcal{U}_1}[\alpha_1]$, respectively

Hence both factorizations are equivalent $\square$

Compare also Hatcher, § I.2.

20. <u>Corollary A</u> Let X be a topological space, $\mathcal{U}$ a collection of open subsets with $X = \bigcup \mathcal{U}$, $p \in \bigcap \mathcal{U}$. Assume that $U \cap V \cap W$ is path connected for all $U, V, W \in \mathcal{U}$ and that $\pi_1(U \cap V; p) = \{[c_p]\}$ is trivial for all $U, V \in \mathcal{U}$ with $U \neq V$. Then

$$\pi_1(X;p) \cong \coprod_{U \in \mathcal{U}} \pi_1(U;p)$$

#

<u>Corollary B</u> (The classical Seifert-Van Kampen Theorem). Let X be a topological space, let $U, V \subseteq X$ be open with $X = U \cup V$ and $p \in U \cap V$. If $U, V, U \cap V$ are path connected, then $\pi_1(X;p)$ is the colimit of the diagram

$$\begin{tikzcd} & \pi_1(U;p) \\ \pi_1(U \cap V;p) \arrow[ur, "(\iota_{V,U})_*"] \arrow[dr, "(\iota_{U,V})_*"'] & \\ & \pi_1(V;p) \end{tikzcd}$$

<u>proof</u> The explicit construction of the colimit in §5.1 shows that it coincides with $\coprod_{i=1,2} \pi_1(U_i;p) / \langle\langle L \rangle\rangle$ $\square$