§ 5. Amalgams and HNN-extensions

Let $(G_i)_{i \in I}$ be a (nonempty) family of grps, let H be a grp, with homomorphism $\varepsilon_i : H \to G_i$. We are interested in the colimit G of the diagram

$$\begin{tikzcd} & G_i \\ H \arrow[ur] \arrow[r] & G_j \\ & \vdots \end{tikzcd}$$

Since grp is cocomplete, such a colimit exists. In the special case $\# I = 2$, this is a push-out,

$$\begin{tikzcd} H \arrow[r] \arrow[d] & G_1 \arrow[d] \\ G_2 \arrow[r] & G \end{tikzcd}$$

1. Construction We let $\varepsilon_i : H \to G_i$ be as above and we consider the free product $\coprod_{i \in I} G_i$, with the canonical homomorphism $\iota_j : G_j \to \coprod_{i \in I} G_i$.

Put $Z = \{ \iota_j \varepsilon_j(h) \iota_k \varepsilon_k(h^{-1}) \mid h \in H, j, k \in I \}$

and $N = \langle\langle Z \rangle\rangle \trianglelefteq \coprod_{i \in I} G_i$

$\pi : \coprod_{i\in I} G_i \longrightarrow \coprod_{i\in I} G_i / N = G$ and

$\bar{\iota}_j = \pi \circ \iota_j : G_j \to G$.

<u>Lemma</u> G with the homomorphisms

$$G_j \xrightarrow{\bar{\iota}_j} G \quad \text{and} \quad H \xrightarrow{\bar{\iota}_j \circ \varepsilon_j} G \quad (\text{any } j)$$

is the colimit of the diagram

$$\begin{tikzcd} & G_i \\ H \arrow[ur, "\varepsilon_i"] \arrow[r, "\varepsilon_j"] \arrow[dr, "\varepsilon_k"'] & G_j \\ & G_k \\ & \vdots \end{tikzcd}$$

pf Let K be any group, with homomorphisms $\varkappa_j : G_j \to K$ such that $\varkappa_j \circ \varepsilon_j = \varkappa_k \circ \varepsilon_k$ for all j,k.

$$\begin{tikzcd} & G_j \arrow[dr, "\varkappa_j"] & \\ H \arrow[ur, "\varepsilon_j"] \arrow[dr, "\varepsilon_k"'] & & K \\ & G_k \arrow[ur, "\varkappa_k"'] & \end{tikzcd}$$

We define $\varkappa : G \to K$ as follows.

Consider $\coprod_{i\in I} G_i \xrightarrow{\tilde{\varkappa}} K$ given by

$\tilde{\varkappa}(g) = \varkappa_j(\iota_j(g))$ if $g \in G_j \subseteq \coprod_{i\in I} G_i$

(universal property of the free product).

For $h \in H$ and $j, k \in I$ we have

$$\tilde{\varkappa}(\iota_j(\varepsilon_j(h)) = \varkappa_j \circ \varepsilon_j(h) = \varkappa_k \circ \varepsilon_k(h) =$$

$$\tilde{\varkappa}(\iota_k(\varepsilon_k(h))\text{, whence } N \subseteq \ker(\tilde{\varkappa}).$$

Hence there is a <u>unique</u> homomorphism

$$\varkappa : G \to K \quad \text{with } \varkappa \circ \pi = \tilde{\varkappa}. \qquad \square$$

This does not tell us much about G.

2. <u>Def</u> Let G be a grp, let H be a subgrp.

A <u>left transversal</u> for H in G is a map

$G \to G$, $g \mapsto \bar{g}$ with the following properties.

(LT1) $\bar{a}H = aH$ for all a

(LT2) $\bar{a}H = \bar{b}H \Leftrightarrow aH = bH$

(LT3) $\bar{e} = e$

$\Leftrightarrow$ $\bar{a}$ is left coset representative for aH

(cp. also Lemma §1.12)

Left transversals exist by the axiom of choice.

Note: $\bar{\bar{a}} = \bar{a}$, $\bar{a}^{-1}a \in H$, $\overline{a\bar{b}} = \overline{ab}$

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3. Let $H \xrightarrow{\varepsilon_i} G_i$ be a nonempty family of group homomorphisms, let $W = \coprod_{i\in I} G_i$ = {reduced words} and put

$Z = \{ \iota_j \varepsilon_j(h)\, \iota_k \varepsilon_k(h^{-1}) \mid h \in H,\ j,k \in I \}$, $N = \langle\langle Z \rangle\rangle \trianglelefteq W$

$G = W/N$ the colimit of the diagram, as in § 5.1. Put $H_i = \varepsilon_i(H) \subseteq G_i$ and choose for each $i \in I$ a left transversal for $H_i \subseteq G_i$.

Put $A = \{ (\bar{a}_1, \dots, \bar{a}_r, h) \mid r \geq 0,\ (\bar{a}_1, \dots, \bar{a}_r) \text{ reduced},\ h \in H \}$

The elements of A are called <u>normal forms</u>.
Choose $o \in I$.

<u>Proposition</u> With the notation set up as above, every coset $gN \in G$ has a representative $(\bar{a}_1, \dots, \bar{a}_r, h) \in A$, such that

$$gN = \bar{a}_1 \bar{a}_2 \cdots \bar{a}_r h_o N \qquad \text{where } h_o = \iota_o \varepsilon_o(h)$$

Be for we prove this we note the following.

Given $h \in H_j$, choose $\tilde{h} \in H$ with $\varepsilon_j(\tilde{h}) = h$. For $k \in I$, put $h' = \varepsilon_k(\tilde{h})$. Then $\iota_j(h)N = \iota_k(h')N$.

pf Let $(x_1, \dots, x_r) \in W$ be a <u>shortest</u> reduced word in the coset $gN = x_1 \cdots x_r N$, $x_j \in G_{i_j} = G_j$

$j = 1, \dots, r$

Put $a_1 = x_1$, and $a_1 = \bar{a}_1 h_1 \quad h_1 \in H_1$

$$\begin{aligned}
x_1 \cdots x_r N &= \bar{a}_1 h_1 x_2 \cdots x_r N \\
&= \bar{a}_1 \underbrace{(h_2' x_2)}_{= a_2} x_3 \cdots x_r N \qquad h_2' \in H_2 \\
&= \bar{a}_1 (\bar{a}_2 h_2) x_3 \cdots x_r N \qquad h_2 \in H_2 \\
&= \bar{a}_1 \bar{a}_2 \underbrace{(h_3' x_3)}_{= a_3} x_4 \cdots x_r N \qquad h_3' \in H_3 \\
&\ \ \vdots \\
&= \bar{a}_1 \cdots \bar{a}_{r-1} a_r N
\end{aligned}$$

Note: $\bar{a}_1, \dots \bar{a}_{r-1}, a_r$ are non trivial by minimality. If $\bar{a}_r \neq e$ put $\bar{a}_r = a_r h_r$

$$\begin{aligned}
gN &= \bar{a}_1 \cdots \bar{a}_r h_r N \\
&= \bar{a}_1 \cdots \bar{a}_r h_0 N \qquad \text{(done)}
\end{aligned}$$

If $\bar{a}_r = e$ put $h_r = a_r$

$$gN = \bar{a}_1 \cdots \bar{a}_{r-1} h_r = \bar{a}_1 \cdots \bar{a}_{r-1} h_0 N \qquad \text{(done)} \qquad \square$$

4. <u>Theorem</u> Let $H \xrightarrow{\varepsilon_i} G_i$ be a (nonempty) family of <u>injective</u> group homomorphisms. With the notation set up in §5.3, every coset $gN \in W/N$ has a <u>unique</u> representative in A.

<u>pf</u> Fix $j \in I$. We first construct an action
$$G_j \times A \to A \text{ as follows.}$$
Let $(\bar{a}_1, \dots, \bar{a}_r, h) \in A$, $a_s \in G_{i_s} = G_s$, $s = 1, \dots, r$

and $g \in G_j$.

$\boxed{\text{Case } j \neq i_1}$ Re-write

$$g\bar{a}_1 \cdots \bar{a}_r h_0 N = (\bar{g} h_j)\bar{a}_1 \cdots \bar{a}_r h_0 N$$
$$= \bar{g}\,\underbrace{(h_1' \bar{a}_1)}_{=b_1}\,\bar{a}_2 \cdots \bar{a}_r h_0 N$$
$$= \bar{g}\, b_1 \bar{a}_2 \cdots a_r h_0 N$$
$$= \bar{g}\,(\bar{b}_1 h_1)\,\bar{a}_2 \cdots \bar{a}_r h_0 N$$
$$= \bar{g}\,\bar{b}_1\,\underbrace{(h_2' \bar{a}_2)}_{=b_2}\,\bar{a}_3 \cdots \bar{a}_r h_0 N$$
$$= \bar{g}\,\bar{b}_1\,(\bar{b}_2 h_2)\,\bar{a}_3 \cdots a_r h_0 N$$

$$\vdots$$

$$= \bar{g}\, \bar{b}_1 \cdots \bar{b}_r\, \tilde{h}_0 N$$

Put
$$L_g(\bar{a}_1, \dots, \bar{a}_r, h) = \begin{cases} (\bar{g}, \bar{b}_1, \dots, \bar{b}_r, \tilde{h}) & \text{if } \bar{g} \neq e \\ (\bar{b}_1, \dots, \bar{b}_r, \tilde{h}) & \text{if } \bar{g} = e \end{cases}$$

Note: $b_s \notin H_s$ since $a_s \notin H_s$ and $b_s = h_s' a_s$, $h_s \in H_s$

Since the ε_i are isomorphisms, this re-writing is well-defined.

$\boxed{\text{Case } j = i_1}$
$$(g\, \bar{a}_1)\bar{a}_2 \cdots \bar{a}_r h_0 N$$
$$= \overline{g a_1}\, h_1 \bar{a}_2 \cdots \bar{a}_r\, h_0 N$$
$$= \overline{g a_1}\, \underbrace{(h_1' \bar{a}_2)}_{= b_2} \bar{a}_3 \cdots \bar{a}_r\, h_0 N$$
$$\vdots$$
$$= \overline{g a_1}\, \bar{b}_2 \cdots \bar{b}_r\, \tilde{h}_0 N$$

again $b_s \notin H_s$, put

$$L_g(\bar{a}_1, \dots, \bar{a}_r, h) = (\overline{g a_1}, \bar{b}_1, \dots, \bar{b}_r, \tilde{h})$$

We claim that this is an action.

Let $x, y, a \in G_k$ put $z = xy$. Then

$$z\bar{a} = \overline{z\bar{a}}\,\underbrace{\overline{z\bar{a}}^{-1}\,z\bar{a}}_{=h_z} = \overline{za}\,h_z$$

$$y\bar{a} = \overline{ya}\,h_y$$

$$x\,\overline{ya} = \overline{x\,ya}\,\underbrace{\overline{x\overline{ya}}^{-1}\,x\overline{ya}}_{=h_x} = \overline{xya}\,h_x$$

$$\Rightarrow \quad xy\bar{a} = x\,\overline{ya}\,h_y = \overline{xya}\,h_x\,h_y \quad , \quad h_z = h_x h_y$$

From this we see that re-writing is compatible with the multiplication in G_j, hence we obtain an action $G_j \times A \to A$, $G_j \to \mathrm{Sym}(A)$ and thus an action

$$W \times A \to A$$
$$(w, v) \mapsto L_w(v)$$

This action has the following properties.

1. If $h \in H$, then $L_{\varepsilon_i(h)} = L_{\varepsilon_j(h)}$ for all i, j.

   In particular, N acts trivially on A, and we get an induced action $W/N \times A \to A$.

2. If $(\bar{a}_1, \dots, \bar{a}_r, h) \in A$, then

   $$L_{\bar{a}_1 \cdots \bar{a}_r h_0}(e) = \bar{a}_1 \cdots \bar{a}_r h_0 \ ,$$

   hence W (and W/N) act transitively

(3) If $w \in W$ fixes $(e) \in A$, then $w \in N$, because there is $n \in N$ and $v = (\bar{a}_1, \ldots, \bar{a}_r, h) \in A$ with $\bar{a}_1 \cdots \bar{a}_r h_0 \cdot n = w$, whence $L_{\bar{a}_1 \cdots \bar{a}_r h_0} = L_w$, whence $(\bar{a}_1, \ldots, \bar{a}_r, h_0) = (e)$.

Thus the map $W/N \to A$ is bijective,

$$wN \mapsto L_w(e) = v = (\bar{a}_1, \ldots, \bar{a}_r, h)$$

and $wN = \bar{a}_1 \cdots \bar{a}_r h_0 N$ $\square$

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5. Conventions Given a family of homomorphisms $H \xrightarrow{\varepsilon_i} G_i$, we write

$$\varinjlim (H \xrightarrow{\varepsilon_i} G_i)_{i \in I} = W/N = G$$

In case that the ε_i are all injective, one writes also

$$\varinjlim (H \xrightarrow{\varepsilon_i} G_i)_{i \in I} = \mathop{\ast_H}_{i \in I} G_i = G$$

and calls this an amalgamated product. In the special case when H is the trivial group, we obtain again the free product. We put

$$\bar{\iota}_k : G_k \to \mathop{\ast_H}_{i\in I} G_i \qquad \bar{\iota}_k = \pi \circ \iota_k$$

$$\pi : W \to W/N = G$$

$$\text{and } \varepsilon : H \to \mathop{\ast_H}_{i\in I} G_i \ , \quad \varepsilon(h) = \pi \circ \iota_k \circ \varepsilon_k$$

(any $k \in I$)

6. <u>Proposition</u> If the homomorphism $\varepsilon_k : H \to G_k$ are all injective, then the $\bar{\iota}_k$ are injective and ε is also injective. We have

$$\mathop{\ast_H}_{i\in I} G_i = \Big\langle \bigcup_{i\in I} \bar{\iota}_i(G_i) \Big\rangle \quad \text{and}$$

$$\bar{\iota}_k(G_k) \cap \Big\langle \bigcup_{i\neq k} \iota_i(G_i) \Big\rangle = \varepsilon(H)$$

<u>Pf</u> This follows from the normal form for the elements in G. □

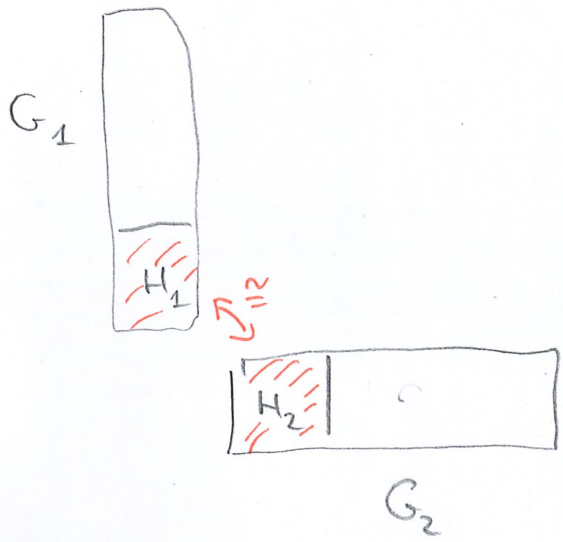
<u>Note:</u> If the ε_k are not injective, this is <u>false</u>.

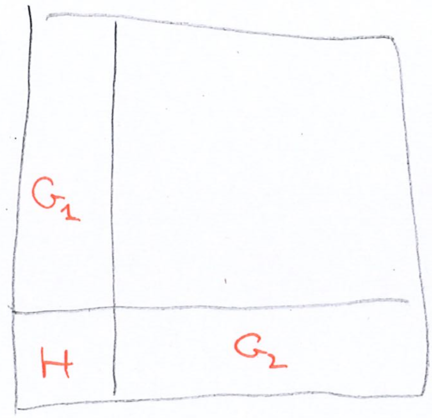
Ex
$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\varepsilon_1} & \mathbb{Z}/2 \\ {\scriptstyle \varepsilon_2}\downarrow & & \downarrow \\ \mathbb{Z}/3 & \longrightarrow & G \end{array} \qquad \text{yields } G = \{1\} \ !$$

$\to$ homework

In view of the proposition, we may view each G_j as a subgrp of $\ast_{H, i \in I} G_i$.

Idea for $I = \{1,2\}$: we glue the grps G_1 and G_2 along the subgrps $H_1 \cong H_2$





7. Prop Let $(H \xrightarrow{\varepsilon_i} G_i)_{i \in I}$ be a family of injective homomorphisms, let $\ast_{H, i \in I} G_i = G$ denote the amalgamated product, let A be the set of normal forms (for a choice of left transversals $H_i \subseteq G_i$). Then we have the following.

(i) If $g \in G$ has normal form
$g \triangleq (\bar{a}_1, \dots, \bar{a}_r, h)$ with $i_1 \neq i_r$, then g has infinite order.

(ii) If there are indices j, k with $j \neq k$ and $H_j \neq G_j$ and $H_k \neq G_k$, then G is infinite.

(iii) if $g \in G$ has finite order, then g is conjugate to an element $\tilde{g} \in G_k \subseteq G$, for some $k \in I$.

pf (i) Consider $(\bar{a}_1, \dots, \bar{a}_r, h) \in A$ and

$$g = \bar{a}_1 \cdots \bar{a}_r h_0 N \in G = W/N$$

Then
$$g^m = \bar{a}_1 \cdots \bar{a}_r h_0 \bar{a}_1 \cdots \bar{a}_r h_0 \cdots N$$
$$= \underbrace{\bar{a}_1 \cdots \bar{a}_r \cdot \underbrace{\overline{h_1 a_1}}_{= \bar{a}_{1,2}} \bar{a}_{2,2} \cdots \bar{a}_{r,2} \bar{a}_{1,3} \cdots \bar{a}_{rm} \tilde{h} N}_{\text{reduced word}}$$

$\Rightarrow g^m \neq e$ for $m \geq 1$.

(ii) Choose $a \in G_j - H_j$, $b \in G_k - H_k$, $j \neq k$

Consider $g \hat{=} (\bar{a}, \bar{b}, e) \in A$ $\leadsto$ g has infinite order.

(iii) Suppose $g \hat{=} (\bar{a}_1, \dots, \bar{a}_r, h)$ has finite order.

$r = 0 \leadsto g \hat{=} h$ $\quad g = h_0 \in H_0 \subseteq G_0$ (✓)

$r = 1 \leadsto g \hat{=} (\bar{a}, h)$ $\quad g = \bar{a} h_1 \in G_1$ (✓)

Now proceed by induction on r.

$g = (\bar{a}_1, \dots, \bar{a}_r, h)$ $\quad r \geq 2$ and $i_1 = i_r$ by (i)

$$\bar{a}_1^{-1} g \bar{a}_1 \hat{=} (\bar{a}_2, \dots, \underbrace{\bar{a}_r h_0 \bar{a}_1}_{= b} N$$

$= \bar{a}_2 \cdots \bar{a}_{r-1} \bar{b} \tilde{h}_0 N \mathrel{\hat{=}} (\bar{a}_2, \dots, \bar{a}_{r-1}, \bar{b}, \tilde{h})$
or $(\bar{a}_2, \dots, \bar{a}_{r-1}, \tilde{h})$ if $\bar{b} = e$

shorter element. $\square$

Corollary If all groups G_i are torsion free (no elts of finite order besides e), then

$$\underset{i \in I}{\ast_H} G_i \text{ is torsion free.}$$

8. HNN - Extensions

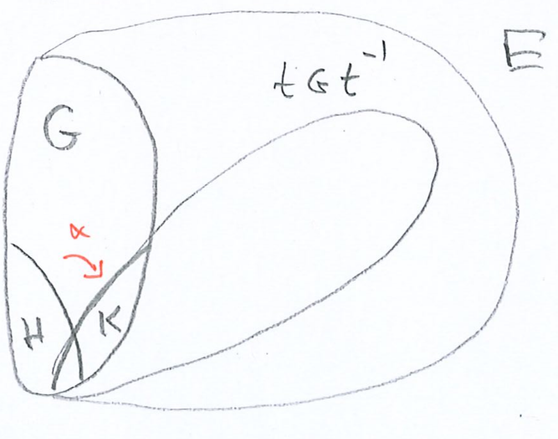
(Graham Higman, Bernhard Neumann, Hanna Neumann)

Let G be a group, with subgroups $K, H \leq G$. Suppose that there is an isomorphism

$$\alpha: H \xrightarrow{\cong} K$$

Then there is a group E, with $G \leq E$ and an element $t \in E$, such that

$$tht^{-1} = \alpha(h) \quad \text{for all } h \in H$$



Pf Choose two letters $u, v \notin G$, put

$U = F(u) \cong \mathbb{Z}$, $V = F(v) \cong \mathbb{Z}$ and put

$X = G * U$, $Y = G * V$. Consider the inclusion

homomorphisms $G \subseteq X$, $G \subseteq Y$.

Now define

$\xi : H \to X$, $h \mapsto u h u^{-1}$

$\eta : K \to Y$, $k \mapsto v k v^{-1}$

which induce homomorphisms

$$\hat{\xi} : G * H \to X \qquad \begin{array}{l} g \mapsto g \\ h \mapsto u h u^{-1} \end{array}$$

$$\hat{\eta} : G * K \to Y \qquad \begin{array}{l} g \mapsto g \\ k \mapsto v k v^{-1} \end{array}$$

Note: $\hat{\xi}$ maps reduced words to reduced words,

eg $\hat{\xi}(g_1, h_1, g_2, h_2, \dots) = g_1 u h_1 u^{-1} g_2 u h_2 u^{-1} \dots$

similarly for $\hat{\eta}$. Hence $\hat{\xi}$ and $\hat{\eta}$ are

injective. Put

$$L = \hat{\xi}(G * H) \subseteq X$$

$$M = \hat{\eta}(G * K) \subseteq Y$$

We obtain an isomorphism

$$\varphi : L \xrightarrow{\cong} \Pi$$

via

$$G * H \xrightarrow{\cong} G * K$$
$$g \longmapsto g$$
$$h \longmapsto \alpha(h)$$

i.e. $\varphi(g) = g$, $\varphi(u h u^{-1}) = v \alpha(h) v^{-1}$

Now we consider the amalgamated product

$X \underset{L}{*} Y$ for

$$\begin{tikzcd} L \arrow[r, phantom, "\subseteq"] \arrow[d, "\varphi"'] & X \arrow[d] \\ Y \arrow[r] & X \underset{L}{*} Y \end{tikzcd}$$

We have an injective homomorphism

$$G \longrightarrow X \longrightarrow X *_L Y$$

In $X *_L Y$ we have $u h u^{-1} = v \alpha(h) v^{-1}$.

If we put $t = v^{-1} u \in X * Y$, then

$$t h t^{-1} = \alpha(h) \text{ in } X *_L Y .$$

Put $E = \langle G \cup \{t\} \rangle \subseteq X *_L Y$ $\square$

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One also writes for short

$$E = G *_\alpha = \langle G, t \mid \alpha(h) = tht^{-1} \text{ for all } h \in H \rangle$$

This is the HNN-Extension of G with respect to α.

Note: if G is torsion free, then E is also torsion free by § 5.7.

HNN-extensions are useful for constructing groups.

9. Theorem (Higman-Neumann-Neumann) Let G be a countably infinite, torsion free group. Then there is a (countable) group U, with $G \subseteq U$, such that all nontrivial elements in U are conjugate: given $v, w \neq e$, there is u such that $v = uwu^{-1}$.

Pf Put $G = \{g_0, g_1, \dots\}$ $g_0 = e$

and $G_0 = G$. We construct groups

$G_0 \subseteq G_1 \subseteq G_2 \subseteq \dots$ such that

$g_1, \dots, g_k$ are conjugate in G_k. Put $G_1 = G_0 = G$. (!)

Suppose that G_k is constructed. Since

$\langle g_k \rangle \cong \mathbb{Z} \cong \langle g_{k+1} \rangle$, we have an isomorphism

$\alpha : \langle g_k \rangle \longrightarrow \langle g_{k+1} \rangle$ $\alpha(g_k) = g_{k+1}$.

Put $G_{k+1} = G_k *_\alpha$ and $G^*_\infty = \bigcup_{k=0}^{\infty} G_k$.

In G^*_∞, all $g_1, g_2, \dots$ are conjugate.

Moreover G^*_∞ is countable and torsion free.

Now put $U_0 = G^*$, $U_1 = (U_0)^*$,

$U_{k+1} = (U_k)^*$ and finally $U = \bigcup_{k=0}^{\infty} U_k$.

$\square$

10. Theorem (H-N-N) Let G be a countable group. Then there is a group K, with $G \leq K$, such that $K = \langle a, t \rangle$ is generated by two elements.

pf Let $a, b \notin G$, $a \neq b$, and $F = F(\{a,b\})$ and $G = \{g_0 = e, g_1, \dots\}$. Put

$$A = \langle \{a, bab^{-1}, b^2ab^{-2}, b^3ab^{-3}, \dots\} \rangle \leq F \leq G * F$$

Claim A is free, with basis $\{a, bab^{-1}, \dots\}$

and $B = \langle \{bg_0, aba^{-1}g_1, a^2ba^{-2}g_2, \dots\} \rangle \leq G * F$ is free with basis $\{bg_0, aba^{-1}g_1, \dots\}$ by §3.10

pf of the claim

For A: consider $(b^{n_1}ab^{-n_1})^{l_1}(b^{n_2}ab^{-n_2})^{l_2} \cdots (b^{n_r}ab^{-n_r})^{l_r}$

$$= b^{n_1}a^{l_1}\underbrace{b^{n_2-n_1}}_{n_1 \neq n_2}a^{l_2} \cdots b^{n_r-n_{r-1}}a^{l_r}b^{-n_r}$$

as reduced word in $F(\{a,b\})$ □

similarly for B

Consider the isomorphism $\alpha : A \xrightarrow{\cong} B$

$$\alpha(b^n a b^{-n}) = a^n b a^{-n} g_n$$

and the HNN-extension $E = (G *_{\alpha} F)*_{\alpha}$

Put $K = \langle \{a, t\} \rangle \leq E$. We have

$b g_0 = b = tat^{-1} \in K$, hence $t, a, b \in K$ for all

$\Rightarrow g_n \in K$ for all n (and $K = E$) $\square$

11. Theorem (B. Neumann) There are $2^{\aleph_0}$ many non-isomorphic groups with two generators.

Pf Put $\mathbb{P} = \{p \in \mathbb{N} \mid p$ prime number$\}$, for $S \subseteq \mathbb{P}$ let $A_S = \bigoplus_{p \in S} \mathbb{Z}/p$. Then A_S is countable abelian, and A_S contains an elmt of order $p \in \mathbb{P}$ iff $p \in S$. Consider the previous construction. The grp $A_S * F(\{a,b\})$ contains elmts of order p iff $p \in S$, because $F(\{a,b\})$ is torsion free, cp. §5.7.

The HNN extension constructed in the previous proof is

$$E = (G * F)*_{\alpha}, \quad G = A_S$$

$$K \leq E \leq (G * F * \mathbb{Z}) *_L (G * F * \mathbb{Z})$$

$$K = \{a, t\}$$

hence E contains elements of order p iff $p \in S$. In other words

$$S = \{ p \in \mathbb{P} \mid K \text{ contains elements of order } p \}$$

Now $\# 2^{\mathbb{P}} = 2^{\aleph_0}$ $\square$