

§4 Limits, colimits, functors

68

We recall some background from category theory

(→ S. MacLane, Categories for the working mathematician)

1. Def A category \mathcal{C} consists of the following data.

(1) A collection of objects, $\text{obj}(\mathcal{C})$

(2) A collection of arrows or morphisms, $\text{mor}(\mathcal{C})$

Every morphism α has a source or domain
 $a = s(\alpha) \in \text{obj}(\mathcal{C})$ and a target or codomain
 $b = t(\alpha) \in \text{obj}(\mathcal{C})$. We depict this as

$$a \xrightarrow{\alpha} b$$

For every object a , there is a special
 morphism id_a , with $s(\text{id}_a) = t(\text{id}_a) = a$.

(3) If α, β are morphisms, with $t(\alpha) = s(\beta)$,

then there is a new morphism $\beta \circ \alpha$, with

$$\begin{aligned} s(\beta \circ \alpha) &= s(\alpha) \\ t(\beta \circ \alpha) &= t(\beta) \end{aligned}$$

$$\circ \xrightarrow{\alpha} \circ \xrightarrow{\beta} \circ$$

We require the following axioms:

(a) put $a = s(\alpha)$, $b = t(\alpha)$. Then

$$\alpha \circ \text{id}_a = \alpha = \text{id}_b \circ \alpha$$

(b) if α, β, γ are morphisms with $t(\alpha) = s(\beta)$
 $t(\beta) = s(\gamma)$

$$\text{then } \gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha$$

Examples

(1) set, the category of sets. Its objects are (all) sets, its morphisms are maps

(2) grp, the category of groups. Its objects are (all) groups, its morphisms are group homomorphisms

(3) Suppose that (P, \leq) is a poset (a partially ordered set). Let \underline{P} be the category whose objects are the elements of P and whose morphisms are pairs (a, b) , with $a \leq b$. Put $\text{id}_a = (a, a)$ and

$$(a, b) \circ (b, c) = (a, c) \quad \begin{array}{l} \text{s}(a, b) = a \\ \text{t}(a, b) = b \end{array} \quad \#$$

(4) Let G be a group. Let \underline{G} be the category with one object $*$. The morphisms are the group elements, with $\text{s}(g) = \text{t}(g) = *$ and composition is multiplication.

(5) Let \mathcal{C} be a category. The opposite category \mathcal{C}^{opp} has the same objects

and morphisms. If α is an arrow in \mathcal{C} ,
we denote by α' the corresponding arrow in \mathcal{C}^{opp} .

We define $t(\alpha') = s(\alpha)$
 $s(\alpha') = t(\alpha)$

and $\alpha' \circ \beta' = (\beta \circ \alpha)'$

so \mathcal{C}^{opp} is a category (!)

Remark There are obvious set-theoretic issues
with examples (1) and (2) - there is no
set containing all sets, or all groups, etc.
We will ignore these and view categories
just as a convenient (meta-) language.

2. Def Let a, b be objects in the category \mathcal{C} .

We put $mor_{\mathcal{C}}(a, b) = \mathcal{C}(a, b) = \{ \alpha \mid \alpha \text{ arrow } s(\alpha) = a, t(\alpha) = b \}$

The category \mathcal{C} is locally small if

$mor_{\mathcal{C}}(a, b)$ is a set (and not a class) for all a, b .

In all examples above, this is the case.

The category \mathcal{C} is small if its objects and
morphisms are contained in some set (and not a class)

The categories in the example (1) (2) are not small, but (3) and (4) are small.

3. Def Let \mathcal{C}, \mathcal{D} be categories. A functor $F: \mathcal{C} \rightarrow \mathcal{D}$

is a map that assigns to every $a \in \text{obj}(\mathcal{C})$ an object $F(a) \in \text{obj}(\mathcal{D})$, to every morphism $\alpha \in \text{mor}(\mathcal{C})$ a morphism $F(\alpha)$, such that

$$(i) \quad s(F(\alpha)) = F(s(\alpha)) \quad t(F(\alpha)) = F(t(\alpha))$$

$$(ii) \quad F(\beta \circ \alpha) = F(\beta) \circ F(\alpha)$$

$$(iii) \quad F(\text{id}_a) = \text{id}_{F(a)}$$

A cofunctor is a map $F: \mathcal{C} \rightarrow \mathcal{D}$ as above,

but with (i') $s(F(\alpha)) = F(t(\alpha)) \quad t(F(\alpha)) = F(s(\alpha))$

$$(ii') \quad F(\beta \circ \alpha) = F(\alpha) \circ F(\beta)$$

$$(iii') \quad F(\text{id}_a) = \text{id}_{F(a)}$$

or, equivalently, a cofunctor is a functor $\mathcal{C} \rightarrow \mathcal{D}^{\text{opp}}$

Exmply (a) $U: \text{grp} \rightarrow \text{set}$ assigns to a grp G its underlying set G

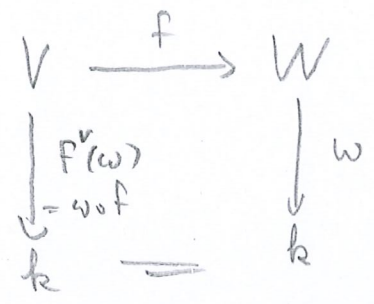
(b) $F: \text{set} \rightarrow \text{grp}$ assigns to a set X the free grp $F(X)$

(c) vect_k the category of k -vector spaces over a field k , with linear maps as morphisms
 \rightarrow cofunctor that assigns to a vector space V

its dual space $V^v = \text{Hom}_K(V, K)$ and to

a linear map $V \xrightarrow{f} W$ its dual f^v ,

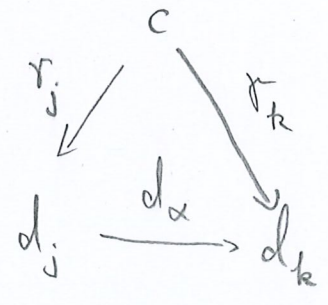
$f^v(w) = w \circ f$



4. Def Let \mathcal{C} be a category. A diagram in \mathcal{C} is a functor $J \xrightarrow{d} \mathcal{C}$, where J is a small category. The diagram is finite if J is finite. For $j \in \text{obj}(J)$ we write $d_j \in \text{obj}(\mathcal{C})$ and for $\alpha \in \text{mor}(J)$ we write $d_\alpha \in \text{mor}(\mathcal{C})$

(Hence a diagram consists of a set of arrows and objects in \mathcal{C} .)

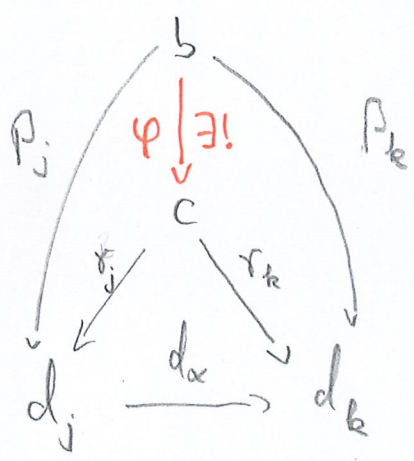
A cone over a diagram is an object c in \mathcal{C} with arrows $c \xrightarrow{\tau_j} d_j$ for all $j \in \text{obj}(J)$, such that we have commuting triangles



for all j, k and all $j \xrightarrow{\alpha} k$

i.e. $\tau_k = d_\alpha \circ \tau_j$

A cone c is called a limit of the diagram if it is minimal in the following sense. For every other cone b , with arrows β_j , there is a unique arrow $b \xrightarrow{\varphi} c$ such that every β_j factors as $\beta_j = \gamma_j \circ \varphi$

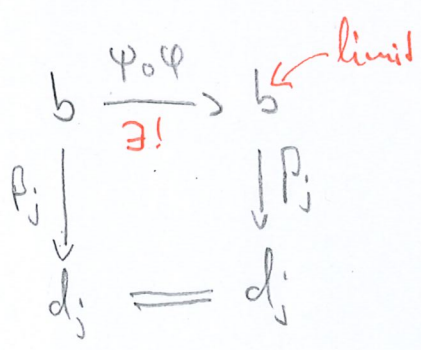


Lemma Limits are unique up to isomorphism. ^(*)

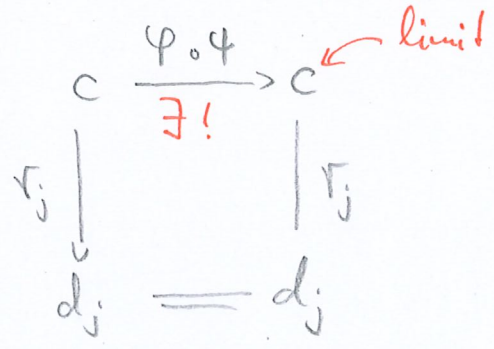
PF Suppose that $c, \{\gamma_j \mid j \in \text{obj}(J)\}$ and $b, \{\beta_j \mid j \in \text{obj}(J)\}$ are both limits.

Hence we have unique arrows $b \xrightarrow{\varphi} c \xrightarrow{\psi} b$ such that $\gamma_j \circ \varphi = \beta_j$ and $\beta_j \circ \psi = \gamma_j$, where

$\beta_j \circ \varphi \circ \psi = \beta_j$ and $\gamma_j \circ \psi \circ \varphi = \gamma_j$. Hence



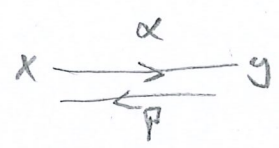
$\Rightarrow \varphi \circ \psi = \text{id}_b$



$\psi \circ \varphi = \text{id}_c$



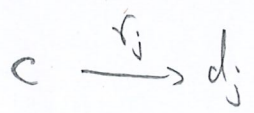
* An isomorphism α is an arrow that has a
 †w-sided inverse arrow β , i.e. $\beta \circ \alpha = id_x$
 $\alpha \circ \beta = id_y$
 for $s(\alpha) = x$
 $t(\alpha) = y$



6. Examples

(1) \mathcal{J} is a discrete category, i.e. \mathcal{J} has no
 morphism besides id_j , for all $j \in \text{obj}(\mathcal{J})$.

Thus a cone consists of any arrows.

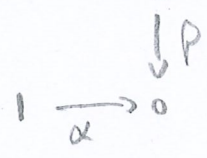


In set, the limit is the cartesian product,

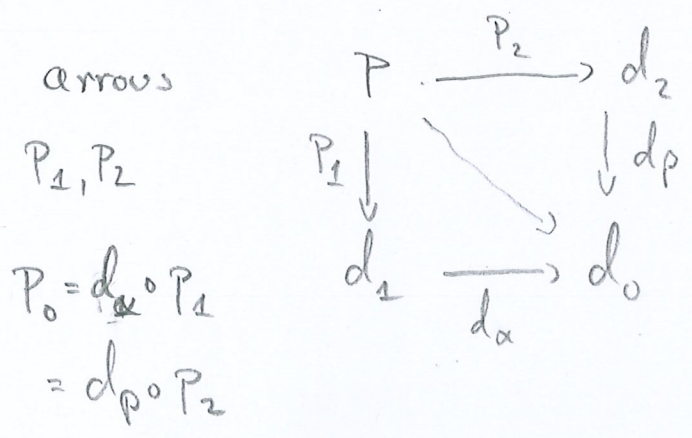
$$C = \prod_{j \in \text{obj}(\mathcal{J})} d_j, \text{ with } \gamma_j = \text{pr}_j$$

In grp, this is also the limit.

(2) \mathcal{J} has three objects $0, 1, 2$ and non-trivial
 arrows



The limit is called a pullback, with



In set, the pullback is

$$P = \{ (x, y) \in d_1 \times d_2 \mid d_\alpha(x) = d_\rho(y) \} \subseteq d_1 \times d_2$$

with $P_1(x, y) = x$

$P_2(x, y) = y$

In grp the pullback is as above.

(3) \mathcal{J} has two objects $0, 1$ and two natural

arrows $0 \begin{array}{c} \xrightarrow{\alpha} \\ \xrightarrow{\rho} \end{array} 1$

The limit is called an equation, with

one arrow $E \xrightarrow{u} d_0$

$$\begin{array}{ccc} & E & \\ u \swarrow & & \searrow d_\alpha u = d_\rho u \\ d_0 & \begin{array}{c} \xrightarrow{d_\alpha} \\ \xrightarrow{d_\rho} \end{array} & d_1 \end{array}$$

In set, the equation is

$$E = \{ z \in d_0 \mid d_\alpha(z) = d_\rho(z) \}, \text{ with } u(z) = z$$

In grp we have again the same equation.

#

7. Def A category \mathcal{C} is complete (finitely complete) if every diagram (every finite diagram) has a limit.

Prop A category \mathcal{C} is complete if and only if it has equalizers and products. It is finitely complete if and only if it has equalizers and finite products.

Pf Let $\mathcal{J} \xrightarrow{d} \mathcal{C}$ be a diagram (resp. a finite diagram). For $\alpha \in \text{mor}(\mathcal{J})$ put $\alpha_0 = d_{\text{src}(\alpha)}$
 $\alpha_1 = d_{\text{tgt}(\alpha)}$

$$\alpha_0 \xrightarrow{\alpha} \alpha_1$$

and $p = \prod_{j \in \text{obj}(\mathcal{J})} d_j$ $q = \prod_{\alpha \in \text{mor}(\mathcal{J})} \alpha_1$ the categorical products!

For $\alpha \in \text{mor}(\mathcal{J})$ we define

$$p \xrightarrow{f_\alpha} \alpha_1 \qquad p \xrightarrow{g_\alpha} \alpha_1$$

via $f_\alpha = \text{pr}_{\alpha_1}$ and $g_\alpha = d_\alpha \circ \text{pr}_{\alpha_0}$
 \uparrow categorical \uparrow categorical

From the universal property of q we obtain unique

arrows $p \xrightarrow{f} q$ such that $\text{pr}_\alpha^f = f_\alpha$
 $\text{pr}_\alpha^g = g_\alpha$

Let $e \xrightarrow{u} p \begin{matrix} \xrightarrow{f} \\ \xleftarrow{g} \end{matrix} q$ be an equalizer of f and g . We put $t_j = pr_j \circ u$

$e \xrightarrow{t_j} d_j$. We claim that this is a limit for \mathcal{I} .

For $\alpha \in \text{mor}(\mathcal{I})$ with $kl(\alpha) = \alpha_0$ $j = \alpha_1$ we have

$$\begin{aligned} d_\alpha \circ t_i &= d_\alpha \circ pr_i \circ u = g_\alpha \circ u = pr_\alpha \circ g \circ u \\ &= pr_\alpha \circ f \circ u = f_\alpha \circ u = pr_{\alpha_1} \circ u = t_j \end{aligned}$$

hence e is a cone. Suppose we have another

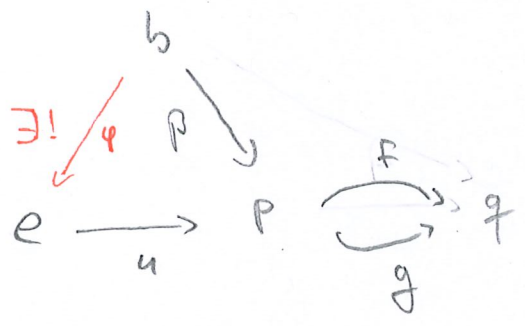
cone b , with arrows $b \xrightarrow{\beta_j} d_j$. Then we obtain a unique arrow $b \xrightarrow{\beta} p$ with $\beta_j = pr_j \circ \beta$.

We have, for $i \xrightarrow{\alpha} j$, $\beta_j = d_\alpha \circ \beta_i$ (because b is a cone). Now we have for all α

$$\begin{aligned} pr_\alpha \circ g \circ \beta &= g_\alpha \circ \beta = d_\alpha \circ pr_{\alpha_0} \circ \beta = d_\alpha \circ \beta_{\alpha_0} \\ &= \beta_{\alpha_1} = pr_{\alpha_1} \circ \beta = f_\alpha \circ \beta = pr_\alpha \circ f \circ \beta, \end{aligned}$$

where $g \circ \beta = f \circ \beta$. Hence this is a unique

morphism $b \xrightarrow{\varphi} e$ with $\beta = u \circ \varphi$



Then $\beta_i = pr_i \circ P = pr_i \circ u \circ \varphi = t_i \circ \varphi$

Conversely, if $b \xrightarrow{\varphi} e$ is an arrow with

$P_i = t_i \circ \varphi$ for all i , then $\beta_i = pr_i \circ u \circ \varphi$

for all i , where $P = u \circ \varphi$, where $\varphi = \varphi$. □

8. Corollary The categories set and grp are complete. (and finitely complete).
The category of finite grps is finitely complete.

9. Suppose that $\mathcal{C} \xrightarrow{F} \mathcal{D}$ is a functor and that $\mathcal{J} \xrightarrow{d} \mathcal{C}$ is a diagram. Then $F \circ \mathcal{J}: \mathcal{J} \rightarrow \mathcal{D}$ is also a diagram. If c is a cone for d , then $F(c)$ is a cone for $F \circ d$. In general, a

Def We call a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ continuous if it preserves limits, i.e. if for every limit c of a diagram $\mathcal{J} \xrightarrow{d} \mathcal{C}$, $F(c)$ is a limit for $F \circ d: \mathcal{J} \rightarrow \mathcal{D}$.

(We call F finitely continuous if this holds for all finite diagrams.)

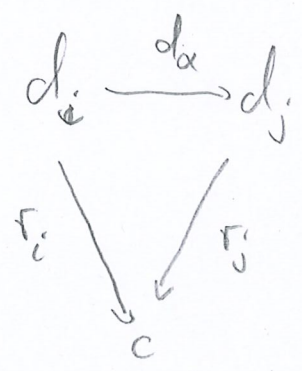
Lemma If a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ (finite products) preserves products and equalizers, then it is continuous (finitely continuous).

PF Follows from (the proof of) Prop. §4.7. □

Corollary The forgetful functor $U: \text{grp} \rightarrow \text{set}$ is continuous. □

Now we define.

10. Def let $J \xrightarrow{d} \mathcal{C}$ be a diagram. A cocone or cone under d is an object $c \in \text{obj}(\mathcal{C})$ with arrows $d_j \xrightarrow{\tau_j} c$, for $j \in \text{obj}(J)$, such that all subdiagrams



commute, $\tau_j \circ d_\alpha = \tau_i$ whenever $i \xrightarrow{d} j$ in J .

A cocone is a colimit if it has the following universal property. If b is any other colimit, with arrows $d_j \xrightarrow{\beta_j} b$, then there is a unique arrow $\varphi: c \rightarrow b$ such that

$$\tau_j \circ \varphi = \beta_j \quad \text{for all } j \in \text{obj}(J).$$

We note that a cocomma in \mathcal{C} is the same as a comma in \mathcal{C}^{opp} . Correspondingly to the examples

§4.6, we have categorical coproducts, push-outs (duals of pullbacks) and coequalizers. A category \mathcal{C} is cocomplete if all colimits in \mathcal{C} exist. By §4.7 and duality, this holds if and only if \mathcal{C} has coproducts and coequalizers.

Example (a) Coproducts in set.

Let \mathcal{J} be a disjoint category. The coproduct of the family $\{d_j \mid j \in \text{obj}(\mathcal{J})\}$ is the set

$\bigsqcup_{j \in \text{obj}(\mathcal{J})} d_j$ (disjoint union of the sets d_j),
with arrows $d_k \xrightarrow{\iota_k} \bigsqcup_{j \in \text{obj}(\mathcal{J})} d_j$ as inclusions.

(b) Coproducts in grp

The coproduct of groups

$G = \bigsqcup_{j \in \text{obj}(\mathcal{J})} G_j$, together with the maps

$\iota_k: G_k \rightarrow G$

has the required property by §2.5

We note that the forgetful functor $U: \mathbf{grp} \rightarrow \mathbf{set}$ (8)
 does not preserve the coproduct, because

$\coprod_{j \in \text{Obj}(J)} G_j$ has not the disjoint union of the groups G_j as its underlying set.

(c) Let $X \begin{matrix} \xrightarrow{\alpha} \\ \xrightarrow{\beta} \end{matrix} Y$ be two maps in set.

Let \sim be the equivalence relation on Y generated by the relation $\alpha(x) \sim \beta(x)$, for $x \in X$.

Then $Y \xrightarrow{q} Y/\sim$ $y \mapsto [y]_{\sim}$ is the coequalizer

(d) Let $G \begin{matrix} \xrightarrow{\alpha} \\ \xrightarrow{\beta} \end{matrix} K$ be two grp homomorphisms.

Put $N = \langle \langle \alpha(g) \cdot \beta(g)^{-1} \mid g \in G \rangle \rangle \trianglelefteq K$

The $\pi: K \rightarrow K/N$ is the coequalizer in grp. (Again, this is diff from (c))

Cor. The category set and grp are both complete and cocomplete.

Now we discuss adjoint functors. We first consider two examples.

II. Examples of adjoint functors

(a) Let $U: \text{Vect}_k \rightarrow \text{Set}$ be the functor that assigns to a vector space V its underlying set V and let $k[-]: \text{Set} \rightarrow \text{Vect}_k$ be the functor that assigns to a set X the vector space $k[X]$ whose vectors are formal (finite) linear combinations of elements of X .

We have bijections, for a set X and a vector space W

$$\text{Mor}_{\text{Vect}_k}(k[X], W) \cong \text{Mor}_{\text{Set}}(X, U(W))$$

\uparrow
linear maps
 \uparrow
maps

$$\begin{array}{ccc}
 f & \longmapsto & f^\# = f|_X \\
 \varphi^\flat & \longleftarrow & \varphi
 \end{array}$$

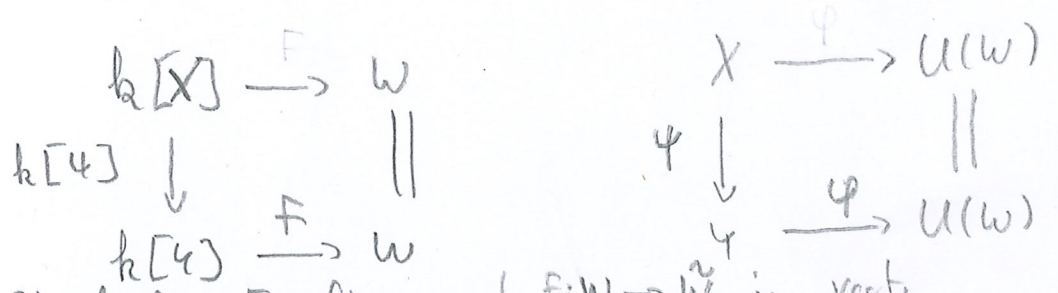
\uparrow linear map defined by extending φ linearly

If $X \xrightarrow{\varphi} Y$ is a map of sets, we have a linear map $k[\varphi]: k[X] \rightarrow k[Y]$ by extending linearly, and the diagram

$$\begin{array}{ccc}
 \text{Mor}_{\text{Vect}_k}(k[X], W) & \xrightarrow{\#} & \text{Mor}_{\text{Set}}(X, U(W)) \\
 \uparrow k[\varphi]^* & & \uparrow \varphi^* \\
 \text{Mor}_{\text{Vect}_k}(k[Y], W) & \xrightarrow{\#} & \text{Mor}_{\text{Set}}(Y, U(W))
 \end{array}$$

commutes, for $\varphi^*(\varphi) = \varphi \circ \varphi$

$$k[\varphi]^*(F) = F \circ k[\varphi] \circ f$$



Similarly for linear maps $F: W \rightarrow W$ in Vect_k

(b) $U: \text{grp} \rightarrow \text{set}$, $U(G)$ underlying set of the grp G

$F: \text{set} \rightarrow \text{grp}$, $F(X)$ free grp on X

$$\text{mor}_{\text{grp}}(F(X), K) \xrightleftharpoons[\text{b}]{\#} \text{mor}_{\text{set}}(X, U(K))$$

$$F^\# = F|_X \quad \text{for a homomorphism } F: F(X) \rightarrow K$$

For a map $X \xrightarrow{\varphi} U(K)$ we have

$$\varphi^\flat: F(X) \rightarrow K \quad (\text{which we denoted by } F(\varphi) \text{ earlier})$$

etc.

12. Def Let $\mathcal{C} \begin{matrix} \xrightarrow{L} \\ \xleftarrow{R} \end{matrix} \mathcal{D}$ be functors.

We say that L and R are adjoint if there are natural bijections

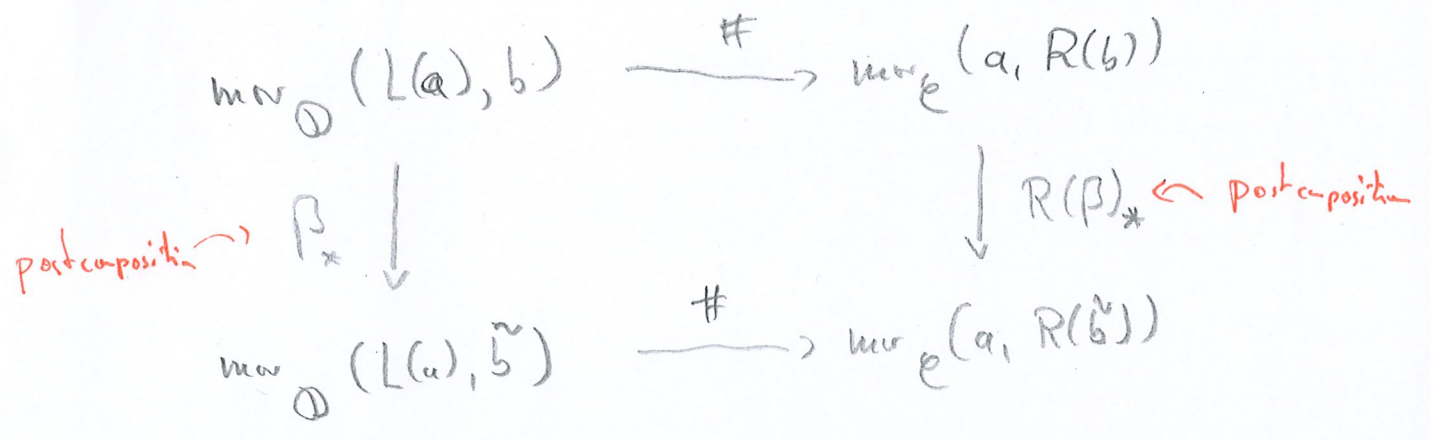
$$\text{mor}_{\mathcal{D}}(L(a), b) \xrightarrow{\#} \text{mor}_{\mathcal{C}}(a, R(b))$$

for all $a \in \text{obj}(\mathcal{C})$ and $b \in \text{obj}(\mathcal{D})$.

The functor L is left adjoint and the functor R is right adjoint.

Natural bijections means that the following diagrams commute.

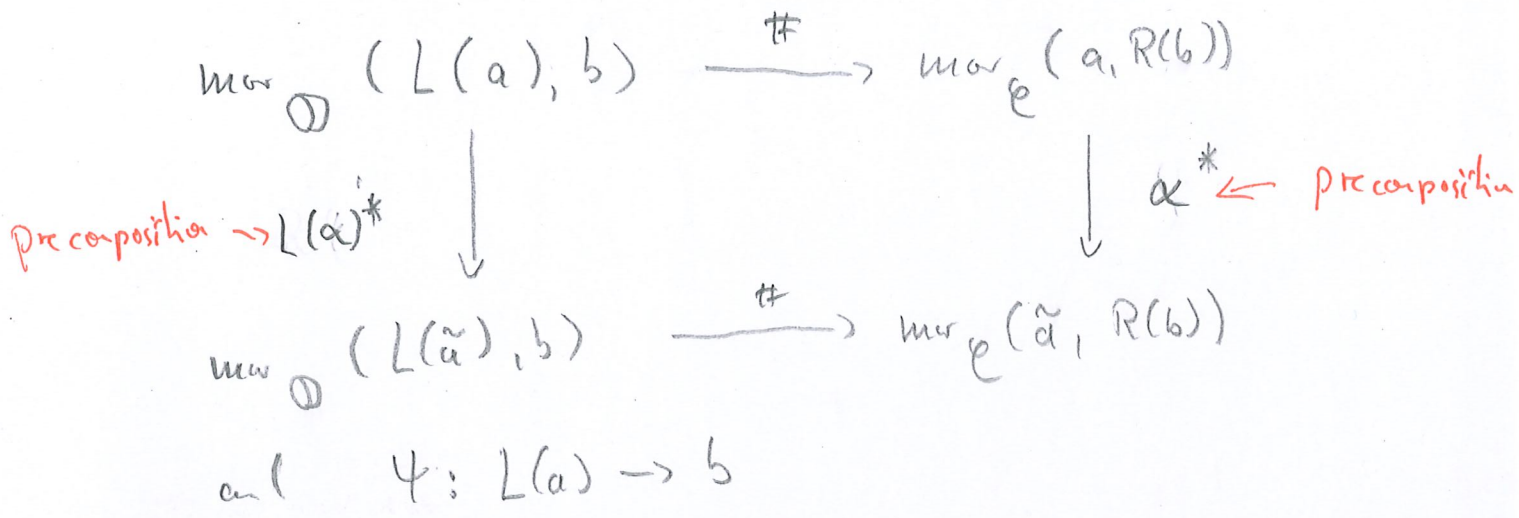
(1) for $b \xrightarrow{\beta} \tilde{b}$



$$L(a) \xrightarrow{\varphi} b$$

$$\begin{aligned}
 & (\beta_*(\varphi))^\# = R(\beta)_*(\varphi^\#) \\
 \Leftrightarrow & (\beta \circ \varphi)^\# = R(\beta) \circ \varphi^\#
 \end{aligned}$$

and (2) for $\tilde{a} \xrightarrow{\downarrow} a$

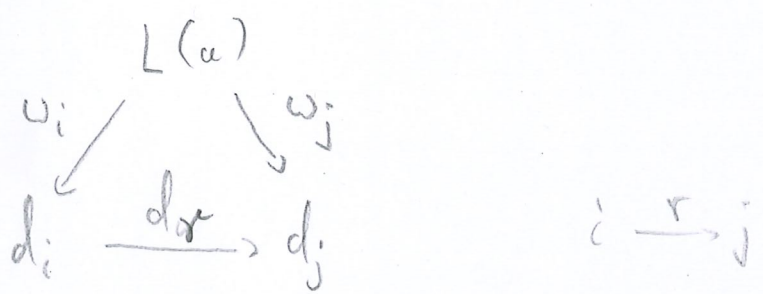


$$\begin{aligned}
 (L(\alpha)^*(\psi))^{\#} &= \alpha^*(\psi^{\#}) \\
 \Leftrightarrow (\psi \circ L(\alpha))^{\#} &= \psi^{\#} \circ \alpha
 \end{aligned}$$

What is this good for?

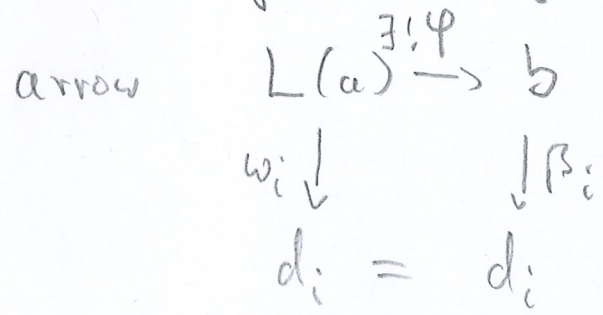
13. Proposition Suppose $\text{Hom } \mathcal{C} \xrightleftharpoons[R]{L} \mathcal{D}$ are adjoint functors. Then R is continuous and L is cocontinuous.

P.F Let $J \xrightarrow{d} \mathcal{D}$ be a diagram, with a limit $b \xrightarrow{\beta_j} d_j$, $j \in \text{Obj}(J)$. The $R(b) \xrightarrow{R(\beta_j)} R(d_j)$, $j \in \text{Obj}(J)$ is a cone over $R \circ d: J \rightarrow \mathcal{C}$. Let $\alpha \xrightarrow{\alpha_j} R(d_j)$ be any other cone over $R \circ d$, write $\alpha_j = \omega_j^\#$, $L(\alpha) \xrightarrow{\omega_j} d_j$. Consider



$$(d_{ij} \circ \omega_i)^\# = R(d_{ij}) \circ \omega_i^\# = R(d_{ij}) \circ \alpha_i = \alpha_j = \omega_j^\#$$

whence $d_{ij} \circ \omega_i = \omega_j$. Hence there is a unique



Consider now

$$\begin{array}{ccc}
 a & \xrightarrow{\varphi^\#} & R(b) \\
 \alpha_i \downarrow & & \downarrow R(\beta_i) \\
 R(d_i) & \xrightarrow[\uparrow]{(\downarrow)} & R(d_i)
 \end{array}$$

$R(\beta_i) \circ \varphi^\# = (\beta_i \circ \varphi)^\# = \omega_i^\# = \alpha_i$. If also

$$\begin{array}{ccc}
 a & \xrightarrow{\tilde{\varphi}^\#} & R(b) \\
 \alpha_i \downarrow & & \downarrow R(\beta_i) \\
 R(d_i) & = & R(d_i)
 \end{array}$$

commutes, then

$\omega_i^\# = \alpha_i = R(\beta_i) \circ \tilde{\varphi}^\# = (\beta_i \circ \tilde{\varphi})^\#$, where

$\omega_i = \beta_i \circ \tilde{\varphi}$, where $\tilde{\varphi} = \varphi$ by uniqueness of φ .

Hence $R(b) \xrightarrow{R(\beta_i)} R(d_i)$, $i \in \text{obj}(J)$ is a limit over Rod .

The proof that L is cocontinuous is dual.



14. Examples of adjoint functors

S. Mac Lane "Adjoint functors everywhere"

$$(a) \quad \underline{\text{set}} \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{U} \end{array} \text{grp}$$

F free gp
 U forget group structure

$$(b) \quad \underline{\text{set}} \begin{array}{c} \xrightarrow{FA} \\ \xleftarrow{U} \end{array} \underline{\text{abgrp}}$$

FA free abelian gp
 U forget gp structure

↑ category of abelian gp

$$(c) \quad \underline{\text{grp}} \begin{array}{c} \xrightarrow{ab} \\ \xleftarrow{incl} \end{array} \underline{\text{abgrp}}$$

ab abelianization
 $incl$ subcategory

$$(d) \quad \underline{\text{metr}} \begin{array}{c} \xrightarrow{cpl} \\ \xleftarrow{incl} \end{array} \underline{\text{cmetr}}$$

$metr$ metric spaces + isometric maps
 $cmetr$ complete metric completion
 cpl

$$(e) \quad \underline{\text{tych}} \begin{array}{c} \xrightarrow{\beta} \\ \xleftarrow{incl} \end{array} \underline{\text{comp}}$$

$tych$ $T_{3\frac{1}{2}}$ -spaces
 $comp$ compact spaces
 β Čech-Stone compactification

$$(f) \quad \underline{\text{hgrp}} \begin{array}{c} \xrightarrow{b} \\ \xleftarrow{incl} \end{array} \underline{\text{compgrp}}$$

$hgrp$ Hausdorff topological groups
 $compgrp$ compact topological groups

b Bohr compactification

15. Remark We constructed the free s.p. $F(X)$ with a special map $i_X: X \rightarrow F(X)$ and used this in §2.8 to show that F is a functor. Let us restate this categorically.

We have: a functor $F: \text{set} \rightarrow \text{grp}$

- for every X , a map $X \xrightarrow{i_X} UF(X)$ in set
- for every map $X \xrightarrow{\varphi} U(H)$ in set, a unique homomorphism $F(X) \xrightarrow{\varphi^b} H$ with

$$\begin{array}{ccc}
 X & \xrightarrow{i_X} & UF(X) & & F(X) \\
 & \searrow \varphi & \downarrow U(\varphi^b) & & \exists! \downarrow \varphi^b \\
 & & U(H) & & H \quad \#
 \end{array}$$

This holds in general. Let $\mathcal{C} \xrightleftharpoons[R]{L} \mathcal{D}$ be adjoint functors. For $a \in \text{obj}(\mathcal{C})$, put $i_a = (\text{id}_{L(a)})^\#$,

$$a \xrightarrow{i_a} RL(a) \quad L(a) \xrightarrow{=} L(a)$$

Suppose that $L(a) \xrightarrow{\omega} b$, *then*

$$R(\omega) \circ (\text{id}_{L(a)})^\# = (\omega \circ \text{id}_{L(a)})^\# = \omega^\#$$

whence $\omega^\# = \alpha$ holds iff and only if

$$R(\omega) \circ i_a = \alpha \iff \omega = \alpha^b$$

b inverse of $\#$

$$\begin{array}{ccc}
 \alpha & \xrightarrow{i_a} & RL(a) \\
 & \searrow & \downarrow R(\alpha^b) \\
 \text{given } \rightarrow \alpha & & R(b)
 \end{array}
 \quad
 \begin{array}{ccc}
 & & L(a) \\
 & & \exists! \downarrow \alpha^b \\
 & & b
 \end{array}$$

16. Lemma (Local description of left adjoints)

let $\mathcal{C} \xleftarrow{R} \mathcal{D}$ be a functor, let $L: \text{obj}(\mathcal{C}) \rightarrow \text{obj}(\mathcal{D})$ be a map, and let $i: \text{obj}(\mathcal{C}) \rightarrow \text{mor}(\mathcal{C})$ be a map, with $a \xrightarrow{i_a} RL(a)$.

Suppose that for each $a \xrightarrow{\alpha} R(b)$ in \mathcal{C} there is a unique $L(a) \xrightarrow{\alpha^b} b$ with

$$\alpha = R(\alpha^b) \circ i_a$$

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{i_a} & RL(a) & & L(a) \\ & \searrow \alpha & \downarrow R(\alpha^b) & \exists! \downarrow \alpha^b & \\ & & R(b) & & b \end{array}$$

Then L is part of a functor $\mathcal{C} \xrightarrow{L} \mathcal{D}$ which is left adjoint to R , α^b is the inverse of $\#$,

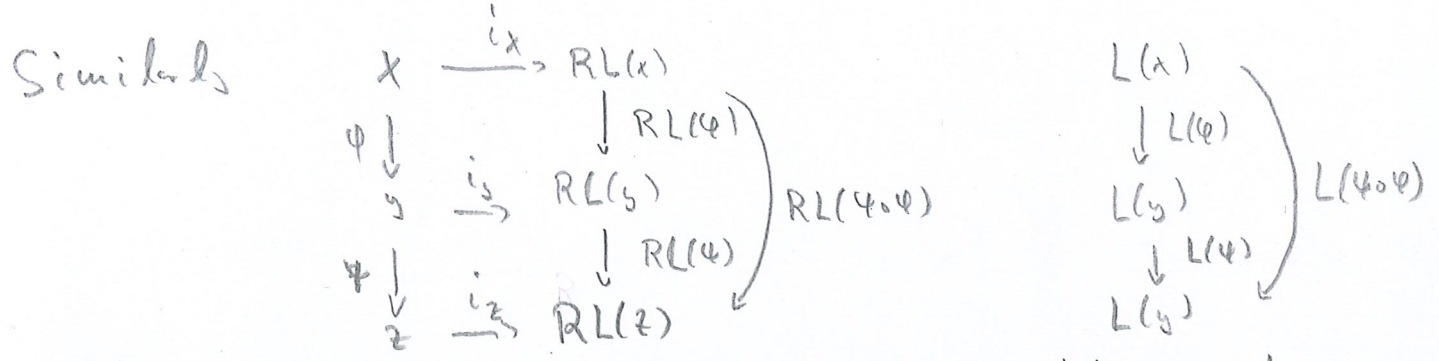
and $L(\varphi) = (i_y \circ \varphi)^b$, for $x \xrightarrow{\varphi} y$ in \mathcal{C} .

PF We define $L(\varphi) = (i_y \circ \varphi)^b$, for $x \xrightarrow{\varphi} y$ in \mathcal{C}

$$\begin{array}{ccc} x & \xrightarrow{i_x} & RL(x) & & L(x) \\ \varphi \downarrow & \searrow & \downarrow RL(\varphi) & \exists! \downarrow (i_y \circ \varphi)^b = L(\varphi) & \\ y & \xrightarrow{i_y} & RL(y) & & L(y) \end{array}$$

If $x=y$ and $\varphi = id_x$, we have necessarily

$$L(\varphi) = (i_x)^b = id_{L(x)} \quad \text{by uniqueness.}$$



Shows that $L(\varphi \circ \varphi) = L(\varphi) \circ L(\varphi)$. Hence L is a functor, $\mathcal{C} \xrightarrow{L} \mathcal{D}$.

For $L(a) \xrightarrow{\alpha} b$ we define $\alpha^\#$ as

$$\alpha^\# = R(\alpha) \circ i_a \quad \left| \quad \begin{array}{l} \text{Thus } (\alpha^\#)^b = \alpha \\ \text{by uniqueness} \end{array} \right.$$

$$a \xrightarrow{i_a} RL(a) \xrightarrow{R(\alpha)} R(b)$$

$$a \text{ and } (\alpha^b)^\# = \alpha \quad \text{because } R(\alpha^b) \circ i_a = \alpha$$

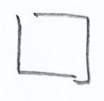
Hence α is a bijection, with inverse $\#$.

For $L(a) \xrightarrow{\varphi} b \xrightarrow{\beta} \tilde{b}$ we have

$$(\beta \circ \varphi)^\# = R(\beta \circ \varphi) \circ i_a = R(\beta) \circ R(\varphi) \circ i_a = R(\beta) \circ \varphi^\#$$

and for $\tilde{a} \xrightarrow{\alpha} a \quad L(a) \xrightarrow{\varphi} b$

$$\begin{aligned}
 (\varphi \circ L(\alpha))^\# &= R(\varphi \circ L(\alpha)) \circ i_{\tilde{a}} = R(\varphi) \circ RL(\alpha) \circ i_{\tilde{a}} \\
 &= R(\varphi) \circ i_a \circ \alpha = \varphi^\# \circ \alpha
 \end{aligned}$$



Remark Freyd's Adjoint Functor Theorem states the following. Let $\mathcal{C} \xleftarrow{R} \mathcal{D}$ be a continuous functor. Assume that \mathcal{D} is complete and that \mathcal{C} and \mathcal{D} are locally small. Then R has a left adjoint $L: \mathcal{C} \rightarrow \mathcal{D}$ if and only if the solution set condition is satisfied:

(*) for every $a \in \text{obj}(\mathcal{C})$ there is a set $(!) I = I(a)$ and a family of morphisms $\alpha_i: a \rightarrow R(b_i)$, such that every morphism $a \xrightarrow{\alpha} R(b)$ factors as

$$\begin{array}{ccc}
 a & \xrightarrow{\alpha} & R(b) \\
 \alpha_i \searrow & & \nearrow R(\beta_i) \\
 & R(b_i) &
 \end{array}
 \quad \text{for some } i \text{ and some } b_i \xrightarrow{\beta_i} b.$$

The theorem can be used to show the existence of

$F: \text{set} \rightarrow \text{grp}$ or $\beta: \text{tych} \rightarrow \text{comp}$ (Čech-Stone)
 Free
 or $b: \text{ltg} \rightarrow \text{cgp}$ (Baker)

It gives, however, no clue about the nature of these functors.