§ 4 Limits, colimits, functors

We recall some background from category theory

(→ S. MacLane, Categories for the working mathematician)

1. Def A category $\mathcal{C}$ consists of the following data.

(1) A collection of objects, $obj(\mathcal{C})$

(2) A collection of arrows or morphisms, $mor(\mathcal{C})$

Every morphism α has a source or domain $a = s(\alpha) \in obj(\mathcal{C})$ and a target or codomain $b = t(\alpha) \in obj(\mathcal{C})$. We depict this as

$$a \xrightarrow{\alpha} b$$

For every object a, there is a special morphism id_a, with $s(id_a) = t(id_a) = a$.

(3) If α, β are morphisms, with $t(\alpha) = s(\beta)$, then there is a new morphism $\beta \circ \alpha$, with

$s(\beta \circ \alpha) = s(\alpha)$
$t(\beta \circ \alpha) = t(\beta)$

$$\bullet \xrightarrow{\alpha} \bullet \xrightarrow{\beta} \bullet$$

We require the following axioms:

(a) put $a = s(\alpha)$, $b = t(\alpha)$. Then

$$\alpha \circ id_a = \alpha = id_b \circ \alpha$$

(b) if α, β, γ are morphisms with $t(\alpha) = s(\beta)$, $t(\beta) = s(\gamma)$

then $\gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha$

Examples

(1) <u>set</u>, the category of sets. Its objects are (all) sets, its morphisms are maps

(2) <u>grp</u>, the category of groups. Its objects are (all) groups, its morphisms are grp homomorphisms

(3) Suppose that $(P, \leq)$ is a <u>poset</u> (a partially ordered set). Let $\underline{P}$ be the category whose objects are the elements of P and whose morphisms are pairs (a,b), with $a \leq b$. Put $id_a = (a,a)$ and

$$(a,b) \circ (b,c) = (a,c) \qquad s(a,b) = a$$
$$t(a,b) = b$$

#

(4) Let G be a group. Let $\underline{G}$ be the category with one object $*$. The morphism are the group elements, with $s(g) = t(g) = *$ and composition is multiplication.

(5) Let $\mathcal{C}$ be a category. The <u>opposite category</u> $\mathcal{C}^{opp}$ has the same objects

and morphisms. If α is an arrow in $\mathcal{C}$, we denote by α' the corresponding arrow in $\mathcal{C}^{opp}$.

We define
$$t(\alpha') = s(\alpha)$$
$$s(\alpha') = t(\alpha)$$

and $\alpha' \circ \beta' = (\beta \circ \alpha)'$

$\leadsto$ $\mathcal{C}^{opp}$ is a category (!)

<u>Remark</u> There are obvious set-theoretic issues with examples (1) and (2) – there is no <u>set</u> containing <u>all</u> sets, or <u>all</u> groups, etc. We will <u>ignore these</u> and view categories just as a convenient (meta-) language.

2. <u>Def</u> Let a, b be objects in the category $\mathcal{C}$. We put $mor_{\mathcal{C}}(a,b) = \mathcal{C}(a,b) = \left\{ \alpha \;\middle|\; \begin{array}{l} \alpha \text{ arrow} \\ s(\alpha) = a,\ t(\alpha) = b \end{array} \right\}$

The category $\mathcal{C}$ is <u>locally small</u> if $mor_{\mathcal{C}}(a,b)$ is a set (and not a class) for all a, b.

In all examples above, this is the case.

The category $\mathcal{C}$ is <u>small</u> if its objects and morphisms are contained in some set (and not a class)

The categories in the example (1) (2) are not small, but (3) and (4) are small.

3. <u>Def</u> Let $\mathcal{C}$, $\mathcal{D}$ be categories. A <u>functor</u> $F: \mathcal{C} \to \mathcal{D}$ is a map that assigns to every $a \in obj(\mathcal{C})$ an object $F(a) \in obj(\mathcal{D})$, to every morphism $\alpha \in mor(\mathcal{C})$ a morphism $F(\alpha)$, such that

(i) $s(F(\alpha)) = F(s(\alpha))$ $\quad t(F(\alpha)) = F(t(\alpha))$

(ii) $F(\beta \circ \alpha) = F(\beta) \circ F(\alpha)$

(iii) $F(id_a) = id_{F(a)}$

A <u>cofunctor</u> is a map $F: \mathcal{C} \to \mathcal{D}$ as above, but with (i^c) $s(F(\alpha)) = F(t(\alpha))$ $\quad t(F(\alpha)) = F(s(\alpha))$

(ii^c) $F(\beta \circ \alpha) = F(\alpha) \circ F(\beta)$

(iii^c) $F(id_a) = id_{F(a)}$

or, equivalently, a cofunctor is a functor $\mathcal{C} \to \mathcal{D}^{opp}$

<u>Examples</u> (a) $U: grp \to set$ assigns to a grp G its underlying set G

(b) $F: set \to grp$ assigns to a set X the free grp $F(X)$

(c) $vect_k$ the category of k-vector spaces over a field k, with linear maps as morphisms

$\rightsquigarrow$ cofunctor that assigns to a vector space V

its dual space $V^\vee = \mathrm{Hom}_k(V,k)$ and to a linear map $V \xrightarrow{f} W$ its dual $f^\vee$,

$f^\vee(\omega) = \omega \circ f$

$$
\begin{tikzcd}
V \arrow[r, "f"] \arrow[d, "f^\vee(\omega)" , "= \omega\circ f"'] & W \arrow[d, "\omega"] \\
k \arrow[r, equal] & k
\end{tikzcd}
$$

4. *Def* Let $\mathcal{C}$ be a category. A *diagram* in $\mathcal{C}$ is a functor $J \xrightarrow{d} \mathcal{C}$, where J is a small category. The diagram is *finite* if J is finite. For $j \in \mathrm{obj}(J)$ we write $d_j \in \mathrm{obj}(\mathcal{C})$ and for $\alpha \in \mathrm{mor}(J)$ we write $d_\alpha \in \mathrm{mor}(\mathcal{C})$

(Hence a diagram consists of a set of arrows and objects in $\mathcal{C}$.)

A *cone over a diagram* is an object c in $\mathcal{C}$ with arrows $c \xrightarrow{\gamma_j} d_j$ for all $j \in \mathrm{obj}(J)$, such that we have commuting triangles

$$
\begin{tikzcd}
 & c \arrow[dl, "\gamma_j"'] \arrow[dr, "\gamma_k"] & \\
d_j \arrow[rr, "d_\alpha"] & & d_k
\end{tikzcd}
$$

for *all* j, k and *all* $j \xrightarrow{\alpha} k$

i.e. $\gamma_k = d_\alpha \circ \gamma_j$

The cone c is called a <u>limit</u> of the diagram if it is minimal in the following sense. For every other cone b, with arrows β_j, there is a <u>unique</u> arrow $b \xrightarrow{\varphi} c$ such that every β_j factors as $\beta_j = \gamma_j \circ \varphi$

$$\begin{tikzcd}
 & b \arrow[d, "\varphi", "\exists!"'] \arrow[ddl, bend right, "\beta_j"'] \arrow[ddr, bend left, "\beta_k"] & \\
 & c \arrow[dl, "\gamma_j"'] \arrow[dr, "\gamma_k"] & \\
d_j \arrow[rr, "d_\alpha"] & & d_k
\end{tikzcd}$$

<u>Lemma</u> Limits are unique up to isomorphism. ⊛

<u>Pf</u> Suppose that $c, \{\gamma_j \mid j \in obj(J)\}$ and $b, \{\beta_j \mid j \in obj(J)\}$ are both limits.

Hence we have <u>unique</u> arrows $b \xrightarrow{\varphi} c \xrightarrow{\psi} b$ such that $\gamma_j \circ \varphi = \beta_j$ $\beta_j \circ \psi = \gamma_j$, whence $\beta_j \circ \psi \circ \varphi = \beta_j$ and $\gamma_j \circ \varphi \circ \psi = \gamma_j$. Hence

$$\begin{tikzcd}
b \arrow[r, "\psi \circ \varphi", "\exists!"'] \arrow[d, "\beta_j"'] & b \arrow[d, "\beta_j"] \\
d_j \arrow[r, equal] & d_j
\end{tikzcd}
\qquad
\begin{tikzcd}
c \arrow[r, "\varphi \circ \psi", "\exists!"'] \arrow[d, "\gamma_j"'] & c \arrow[d, "\gamma_j"] \\
d_j \arrow[r, equal] & d_j
\end{tikzcd}$$

limit　　　　limit

$\Rightarrow \psi \circ \varphi = id_b$　　　　$\varphi \circ \psi = id_c$

□

* An <u>isomorphism</u> α is an arrow that has a two-sided inverse arrow β, i.e. $\beta\circ\alpha = id_x$, $\alpha\circ\beta = id_y$

for $s(\alpha) = x$, $t(\alpha) = y$

$$x \underset{\beta}{\overset{\alpha}{\rightleftarrows}} y$$

6. <u>Examples</u>

(1) J is a <u>discrete category</u>, i.e. J has <u>no</u> morphism besides id_j, for all $j \in obj(J)$.

Thus a cone consists of any arrows.

$$c \xrightarrow{r_j} d_j$$

In <u>set</u>, the limit is the cartesian product,

$$C = \prod_{j\in obj(J)} d_j \quad , \text{ with } \quad r_j = pr_j$$

In grp, this is also the limit.

(2) J has three objects $0,1,2$ and non-trivial arrows

$$\begin{tikzcd} & 2 \arrow[d, "p"] \\ 1 \arrow[r, "\alpha"'] & 0 \end{tikzcd}$$

The limit is called a <u>pull back</u>, with arrows p_1, p_2

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr] & d_2 \arrow[d, "d_p"] \\ d_1 \arrow[r, "d_\alpha"'] & d_0 \end{tikzcd}$$

$$p_0 = d_\alpha \circ p_1 = d_p \circ p_2$$

In *set*, the pull back is

$$P = \{ (x,y) \in d_1 \times d_2 \mid d_\alpha(x) = d_\beta(y) \} \subseteq d_1 \times d_2$$

with $p_1(x,y) = x$

$p_2(x,y) = y$

In *grp* the pull back is as above.

(3) J has two objects $0, 1$ and two nontrivial arrows

$$\begin{tikzcd} 0 \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] & 1 \end{tikzcd}$$

The limit is called an *equalizer*, with one arrow $E \xrightarrow{u} d_0$

$$\begin{tikzcd} & E \arrow[dl, "u"'] \arrow[dr, "d_\alpha \circ u = d_\beta \circ u"] & \\ d_0 \arrow[rr, bend left, "d_\alpha"] \arrow[rr, bend right, "d_\beta"'] & & d_1 \end{tikzcd}$$

In *set*, the equalizer is

$$E = \{ z \in d_0 \mid d_\alpha(z) = d_\beta(z) \}, \text{ with } u(z) = z$$

In *grp* we have again the same equalizer.

7. <u>Def</u> A category $\mathcal{C}$ is <u>complete</u> (<u>finitely complete</u>) if every diagram (every finite diagram) has a limit.

<u>Prop</u> A category $\mathcal{C}$ is complete if and only if it has equalizers and products. It is finitely complete if and only if it has equalizers and finite products.

<u>pf</u> Let $J \xrightarrow{d} \mathcal{C}$ be a diagram (resp. a finite diagram). For $\alpha \in mor(J)$ put $\alpha_0 = d_{s(\alpha)}$, $\alpha_1 = d_{t(\alpha)}$

$$\alpha_0 \xrightarrow{\alpha} \alpha_1$$

and $p = \prod_{j \in obj(J)} d_j$ $\qquad q = \prod_{\alpha \in mor(J)} \alpha_1$ the categorical products!

For $\alpha \in mor(J)$ we define

$$p \xrightarrow{f_\alpha} \alpha_1 \qquad p \xrightarrow{g_\alpha} \alpha_1$$

via $f_\alpha = pr_{\alpha_1}$ and $g_\alpha = d_\alpha \circ pr_{\alpha_0}$

categorical $\qquad$ categorical

From the universal property of q we obtain unique arrows

$$p \overset{f}{\underset{g}{\rightrightarrows}} q$$

such that $pr_\alpha \circ f = f_\alpha$
$pr_\alpha \circ g = g_\alpha$

Let
$$e \xrightarrow{u} p \underset{g}{\overset{f}{\rightrightarrows}} q$$
be an equalizer of f and g. We put $t_j = pr_j \circ u$

$$e \xrightarrow{t_j} d_j$$
. We claim that this is a limit for $\mathcal{J}$.

For $\alpha \in mor(\mathcal{J})$ with $i = \alpha_0$ $j = \alpha_1$ we have

$$d_\alpha \circ t_i = d_\alpha \circ pr_i \circ u = g_\alpha \circ u = pr_\alpha \circ g \circ u$$
$$= pr_\alpha \circ f \circ u = f_\alpha \circ u = pr_{\alpha_1} \circ u = t_j$$

hence e is a cone. Suppose we have another cone b, with arrows $b \xrightarrow{\beta_j} d_j$. Then we obtain a unique arrow $b \xrightarrow{\beta} p$ with $\beta_j = pr_j \circ \beta$. We have, for $i \xrightarrow{\alpha} j$, $\beta_j = d_\alpha \circ \beta_i$ (because b is a cone). Now we have for all α

$$pr_\alpha \circ g \circ \beta = g_\alpha \circ \beta = d_\alpha \circ pr_{\alpha_0} \circ \beta = d_\alpha \circ \beta_{\alpha_0}$$
$$= \beta_{\alpha_1} = pr_{\alpha_1} \circ \beta = f_\alpha \circ \beta = pr_\alpha \circ f \circ \beta,$$

whence $g \circ \beta = f \circ \beta$. Hence there is a unique morphism $b \xrightarrow{\varphi} e$ with $\beta = u \circ \varphi$

$$\begin{tikzcd} & b \arrow[dl, "\exists! \varphi"'] \arrow[d, "\beta"] & \\ e \arrow[r, "u"'] & p \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & q \end{tikzcd}$$

Then $\beta_i = pr_i \circ \beta = pr_i \circ u \circ \varphi = t_i \circ \varphi$

Conversely, if $b \xrightarrow{\psi} e$ is an arrow with $\beta_i = t_i \circ \psi$ for all i, then $\beta_i = pr_i \circ u \circ \psi$ for all i, whence $\beta = u \circ \psi$, whence $\psi = \varphi$. $\square$

8. Corollary The categories set and grp are complete. (and finitely complete).

The category of finite grps is finitely complete.

9. Suppose that $\mathcal{C} \xrightarrow{F} \mathcal{D}$ is a functor and that $J \xrightarrow{d} \mathcal{C}$ is a diagram. Then $F \circ J : J \to \mathcal{D}$ is also a diagram. If c is a cone for d, then $F(c)$ is a cone for $F \circ d$. In general, a

Def We call a functor $F : \mathcal{C} \to \mathcal{D}$ continuous if it preserves limits, i.e. if for every limit c of a diagram $J \xrightarrow{d} \mathcal{C}$, $F(c)$ is a limit for $F \circ d : J \to \mathcal{D}$.

(We call F <u>finitely continuous</u> if this holds for all finite diagrams.)

<u>Lemma</u> If a functor $F: \mathcal{C} \to \mathcal{D}$ preserves products (finite products) and equalizers, (and if $\mathcal{C}$ is complete) then it is continuous (finitely continuous).

<u>Pf</u> Follows from (the proof of) Prop. §9.7. □

<u>Corollary</u> The forgetful functor $U: \underline{grp} \to \underline{set}$ is continuous. □

Now we dualize.

10. <u>Def</u> Let $J \xrightarrow{d} \mathcal{C}$ be a diagram. A <u>cocone</u> or <u>cone under d</u> is an object $c \in \mathrm{obj}(\mathcal{C})$ with arrows $d_j \xrightarrow{r_j} c$, for $j \in \mathrm{obj}(J)$, such that all subdiagrams

$$\begin{tikzcd} d_i \arrow[rr, "d_\alpha"] \arrow[dr, "r_i"'] & & d_j \arrow[dl, "r_j"] \\ & c & \end{tikzcd}$$

commute, $r_j \circ d_\alpha = r_i$ whenever $i \xrightarrow{\alpha} j$ in J.

A cocone is a <u>colimit</u> if it has the following universal property. If b is any other colimit, with arrows $d_j \xrightarrow{\beta_j} b$, then there is a <u>unique</u> arrow $\varphi: c \to b$ such that

$$r_j \circ \varphi = \beta_j \quad \text{for all } j \in \mathrm{obj}(J).$$

We note that a cocone in $\mathscr{C}$ is the same as a cone in $\mathscr{C}^{opp}$. Correspondingly to the examples §4.6, we have <u>categorical coproducts</u>, <u>push-outs</u> (duals of pull backs) and <u>coequalizers.</u> A category $\mathscr{C}$ is <u>cocomplete</u> if all colimits in $\mathscr{C}$ exist. By §4.7 and duality, this holds if and only if $\mathscr{C}$ has coproducts and coequalizers.

<u>Example</u> (a) Coproducts in <u>set</u>.

Let J be a discrete category. The coproduct of the family $\{d_j \mid j \in obj(J)\}$ is the set

$$\dot{\coprod_{j \in obj(J)}} d_j \qquad \text{(disjoint union of the sets } d_j\text{)},$$

with arrows $d_k \xhookrightarrow{\iota_k} \dot{\coprod\limits_{j\in obj(J)}} d_j$ as inclusions.

(b) Coproducts in <u>grp</u>

The <u>coproduct of groups</u>

$G = \coprod\limits_{j \in obj(J)} G_j$, together with the maps

$$\iota_k : G_k \to G$$

has the required property by §2.5

We note that the forgetful functor $U: \underline{grp} \to \underline{set}$ does not preserve the coproduct, because

$\coprod_{j \in obj(J)} G_j$ has not the disjoint union of the groups G_j as its underlying set.

(c) Let $X \overset{\alpha}{\underset{\beta}{\rightrightarrows}} Y$ be two maps in $\underline{set}$.

Let $\sim$ be the equivalence relation on Y generated by the relation $\alpha(x) \sim \beta(x)$, for $x \in X$.

Then $Y \xrightarrow{q} Y/_{\sim}$, $y \mapsto [y]_{\sim}$ is the coequalizer

(d) let $G \overset{\alpha}{\underset{\beta}{\rightrightarrows}} K$ be two grp homomorphisms.

Put $N = \langle\langle \alpha(g) \cdot \beta(g)^{-1} \mid g \in G \rangle\rangle \trianglelefteq K$

Then $\pi : K \to K/_N$ is the coequalizer in $\underline{grp}$. (Again, this is different from (c))

<u>Cor.</u> The categories $\underline{set}$ and $\underline{grp}$ are both complete and cocomplete.

Now we discuss adjoint functors. We first consider two examples.

II. Examples of adjoint functors

(a) Let $U : \underline{\mathrm{vect}}_k \to \underline{\mathrm{set}}$ be the functor that assigns to a vector space V its underlying set V and let $k[-] : \underline{\mathrm{set}} \to \underline{\mathrm{vect}}_k$ be the functor that assigns to a set X the vector space $k[X]$ whose vectors are formal (finite) linear combinations of elements of X.

We have bijections, for a set X and a vector space W #

$$\mathrm{mor}_{\underline{\mathrm{vect}}_k}(\underbrace{k[X],W}_{\text{linear maps}}) \cong \mathrm{mor}_{\underline{\mathrm{set}}}(\underbrace{X, U(W)}_{\text{maps}})$$

$$f \longmapsto f^{\#} = f|_X$$

$$\varphi^b \longleftarrow\!\shortmid\ \varphi$$

φ^b: linear map defined by extending φ linearly

If $X \xrightarrow{\psi} Y$ is a map of sets, we have a linear map $k[\psi] : k[X] \to k[Y]$ by extending linearly, and the diagram

$$\begin{tikzcd}
\mathrm{mor}_{\underline{\mathrm{vect}}_k}(k[X],W) \arrow[r, "\#"] & \mathrm{mor}_{\underline{\mathrm{set}}}(X, U(W)) \\
\mathrm{mor}_{\underline{\mathrm{vect}}_k}(k[Y],W) \arrow[r, "\#"] \arrow[u, "{k[\psi]^*}"] & \mathrm{mor}_{\underline{\mathrm{set}}}(Y, U(W)) \arrow[u, "\psi^*"']
\end{tikzcd}$$

commutes, for $\psi^*(\varphi) = \varphi \circ \psi$

$$k[\psi]^*(F) = F \circ k[\psi]$$

$$\begin{tikzcd} k[X] \arrow[r, "F"] \arrow[d, "{k[\psi]}"'] & W \arrow[d, equal] \\ k[Y] \arrow[r, "F"] & W \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\psi"'] & U(W) \arrow[d, equal] \\ Y \arrow[r, "\varphi"] & U(W) \end{tikzcd}$$

Similarly for linear maps $f : W \to \tilde{W}$ in $\underline{\text{vect}}_k$

(b) $U : \underline{\text{grp}} \to \underline{\text{set}}$, $U(G)$ underlying set of the grp G

$F : \underline{\text{set}} \to \underline{\text{grp}}$, $F(X)$ free grp on X

$$\mathrm{mor}_{\underline{\text{grp}}}(F(X), K) \underset{\flat}{\overset{\#}{\rightleftarrows}} \mathrm{mor}_{\underline{\text{set}}}(X, U(K))$$

$F^\# = F|_X$ for a homomorphism $F : F(X) \to K$

For a map $X \xrightarrow{\varphi} U(K)$ we have

$\varphi^\flat : F(X) \to K$ (which we denoted by $F(\varphi)$ earlier)

etc.

12. Def Let $\mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}$ be functors.
We say that L and R are adjoint if there are natural bijections

$$\mathrm{mor}_{\mathcal{D}}(L(a), b) \xrightarrow{\#} \mathrm{mor}_{\mathcal{C}}(a, R(b))$$

for all $a \in \mathrm{obj}(\mathcal{C})$ and $b \in \mathrm{obj}(\mathcal{D})$.
The functor L is left adjoint and the functor R is right adjoint.

Natural bijections means that the following diagrams commute.

(1) for $b \xrightarrow{\beta} \tilde{b}$

$$\begin{tikzcd} \mathrm{mor}_{\mathcal{D}}(L(a), b) \arrow[r, "\#"] \arrow[d, "\beta_*"'] & \mathrm{mor}_{\mathcal{C}}(a, R(b)) \arrow[d, "R(\beta)_*"] \\ \mathrm{mor}_{\mathcal{D}}(L(a), \tilde{b}) \arrow[r, "\#"] & \mathrm{mor}_{\mathcal{C}}(a, R(\tilde{b})) \end{tikzcd}$$

postcomposition

postcomposition

$L(a) \xrightarrow{\varphi} b$

$$\boxed{\begin{aligned} & (\beta_*(\varphi))^\# = R(\beta)_*(\varphi^\#) \\ \Leftrightarrow\ & (\beta \circ \varphi)^\# = R(\beta) \circ \varphi^\# \end{aligned}}$$

and (2) for $\tilde{a} \xrightarrow{\alpha} a$

$$\begin{tikzcd}
\mathrm{mor}_{\mathcal{D}}(L(a), b) \arrow[r, "\#"] \arrow[d, "L(\alpha)^*"'] & \mathrm{mor}_{\mathcal{C}}(a, R(b)) \arrow[d, "\alpha^*"] \\
\mathrm{mor}_{\mathcal{D}}(L(\tilde{a}), b) \arrow[r, "\#"] & \mathrm{mor}_{\mathcal{C}}(\tilde{a}, R(b))
\end{tikzcd}$$

precomposition $\to L(\alpha)^*$

$\alpha^* \leftarrow$ precomposition

and $\varphi : L(a) \to b$

$$\boxed{\begin{aligned} \left(L(\alpha)^*(\varphi)\right)^{\#} &= \alpha^*(\varphi^{\#}) \\ \Leftrightarrow \quad (\varphi \circ L(\alpha))^{\#} &= \varphi^{\#} \circ \alpha \end{aligned}}$$

What is this good for?

13. <u>Proposition</u> Suppose that $\mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}$ are adjoint functors. Then R is continuous and L is cocontinuous.

<u>pf</u> Let $J \xrightarrow{d} \mathcal{D}$ be a diagram, with a limit $b \xrightarrow{\beta_j} d_j$, $j \in obj(J)$. Then $R(b) \xrightarrow{R(\beta_j)} R(d_j)$, $j \in obj(J)$ is a cone over $R \circ d : J \to \mathcal{C}$. Let $a \xrightarrow{\alpha_j} R(d_j)$ be any other cone over $R \circ d$, write $\alpha_j = \omega_j^{\#}$, $L(a) \xrightarrow{\omega_j} d_j$. Consider

$$\begin{tikzcd}
 & L(a) \arrow[dl, "\omega_i"'] \arrow[dr, "\omega_j"] & \\
d_i \arrow[rr, "d_\gamma"] & & d_j
\end{tikzcd} \qquad i \xrightarrow{\gamma} j$$

$$(d_\gamma \circ \omega_i)^{\#} = R(d_\gamma) \circ \omega_i^{\#} = R(d_\gamma) \circ \alpha_i = \alpha_j = \omega_j^{\#}$$

whence $d_\gamma \circ \omega_i = \omega_j$. Hence there is a unique arrow

$$\begin{tikzcd}
L(a) \arrow[r, "\exists ! \varphi"] \arrow[d, "\omega_i"'] & b \arrow[d, "\beta_i"] \\
d_i \arrow[r, equal] & d_i
\end{tikzcd}$$

Consider now

$$\begin{tikzcd} a \arrow[r, "\varphi^{\#}"] \arrow[d, "\alpha_i"'] & R(b) \arrow[d, "R(\beta_i)"] \\ R(d_i) \arrow[r, phantom, "=" , "(!)"'] & R(d_i) \end{tikzcd}$$

$R(\beta_i) \circ \varphi^{\#} = (\beta_i \circ \varphi)^{\#} = \omega_i^{\#} = \alpha_i$. If also

$$\begin{tikzcd} a \arrow[r, "\tilde{\varphi}^{\#}"] \arrow[d, "\alpha_i"'] & R(b) \arrow[d, "R(\beta_i)"] \\ R(d_i) \arrow[r, phantom, "="] & R(d_i) \end{tikzcd}$$

commutes, then

$\omega_i^{\#} = \alpha_i = R(\beta_i) \circ \tilde{\varphi}^{\#} = (\beta_i \circ \tilde{\varphi})^{\#}$, whence

$\omega_i = \beta_i \circ \tilde{\varphi}$, when $\tilde{\varphi} = \varphi$ by uniqueness of φ.

Hence $R(b) \xrightarrow{R(\beta_i)} R(d_i)$, $i \in \mathrm{obj}(J)$ is a limit over $R \circ d$.

The proof that L is cocontinuous is dual.

$\square$

14. Examples of adjoint functors

S. Mac Lane "Adjoints are everywhere"

(a) $\underline{set} \underset{U}{\overset{F}{\rightleftarrows}} \underline{grp}$

F free grp
U forget group structure

(b) $\underline{set} \underset{U}{\overset{FA}{\rightleftarrows}} \underline{abgrp}$

FA free abelian grp
U forget grp structure

↑ category of abelian grps

(c) $\underline{grp} \underset{incl}{\overset{ab}{\rightleftarrows}} \underline{abgrp}$

ab abelianization
incl subcategory.

(d) $\underline{metr} \underset{incl}{\overset{cplt}{\rightleftarrows}} \underline{cmetr}$

metr metric spaces + isometric maps
cmetr complete metric
cpl completion

(e) $\underline{tych} \underset{incl}{\overset{\beta}{\rightleftarrows}} \underline{comp}$

$\underline{tych}$ $T_{3\frac{1}{2}}$-spaces
$\underline{comp}$ compact spaces
β Čech-Stone compactification

(f) $hgrp \underset{incl}{\overset{b}{\rightleftarrows}} compgrp$

hgrp Hausdorff topological groups
compgrp compact topological groups

b Bohr compactification

15. Remark. We constructed the free grp $F(X)$ with a special map $i_X : X \to F(X)$ and used this in §2.8 to show that F is a functor. Let us restate this categorically.

We have: • a functor $F : \underline{set} \to \underline{grp}$

- for every X, a map $X \xrightarrow{i_X} UF(X)$ in $\underline{set}$
- for every map $X \xrightarrow{\varphi} U(H)$ in $\underline{set}$, a unique homomorphism $F(X) \xrightarrow{\varphi^\flat} H$ with

$$\begin{tikzcd} X \arrow[r, "i_X"] \arrow[dr, "\varphi"'] & UF(X) \arrow[d, "U(\varphi^\flat)"] & F(X) \arrow[d, "\exists!\ \varphi^\flat"] \\ & U(H) & H \end{tikzcd}$$

#

This holds in general. Let $\mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}$ be adjoint functors. For $a \in obj(\mathcal{C})$, put $i_a = (id_{L(a)})^\#$,

$$a \xrightarrow{i_a} RL(a) \qquad L(a) \xrightarrow{=} L(a)$$

Suppose that $L(a) \xrightarrow{\omega} b$, then

$$R(\omega)\circ(id_{L(a)})^\# = (\omega \circ id_{L(a)})^\# = \omega^\#$$

whence $\omega^\# = \alpha$ holds iff and only if

$$R(\omega)\circ i_a = \alpha \iff \omega = \alpha^\flat$$

$\flat$ inverse of $\#$

$$\begin{tikzcd} a \arrow[r, "i_a"] \arrow[dr, "\alpha"'] & RL(a) \arrow[d, "R(\alpha^\flat)"] & L(a) \arrow[d, "\exists!\ \alpha^\flat"] \\ & R(b) & b \end{tikzcd}$$

given → α

16. **Lemma** (Local description of left adjoints)

Let $\mathcal{C} \xleftarrow{R} \mathcal{D}$ be a functor, let $L : \mathrm{obj}(\mathcal{C}) \to \mathrm{obj}(\mathcal{D})$ be a map, and let $i : \mathrm{obj}(\mathcal{C}) \to \mathrm{mor}(\mathcal{C})$ be a map, with $a \xrightarrow{i_a} RL(a)$.

Suppose that for each $a \xrightarrow{\alpha} R(b)$ in $\mathcal{C}$ there is a unique $L(a) \xrightarrow{\alpha^\flat} b$ with

$$\alpha = R(\alpha^\flat) \circ i_a ,$$

$$\begin{tikzcd} a \arrow[r, "i_a"] \arrow[dr, "\alpha"'] & RL(a) \arrow[d, "R(\alpha^\flat)"] & L(a) \arrow[d, "\exists!"', "\alpha^\flat"] \\ & R(b) & b \end{tikzcd}$$

Then L is part of a functor $\mathcal{C} \xrightarrow{L} \mathcal{D}$ which is left adjoint to R, $\flat$ is the inverse of $\#$, and $L(\varphi) = (i_y \circ \varphi)^\flat$, for $x \xrightarrow{\varphi} y$ in $\mathcal{C}$.

<u>pf</u> We define $L(\varphi) = (i_y \circ \varphi)^\flat$, for $x \xrightarrow{\varphi} y$ in $\mathcal{C}$

$$\begin{tikzcd} x \arrow[r, "i_x"] \arrow[d, "\varphi"'] \arrow[dr] & RL(x) \arrow[d, "RL(\varphi)"] & L(x) \arrow[d, "\exists!"', "(i_y \circ \varphi)^\flat = L(\varphi)"] \\ y \arrow[r, "i_y"] & RL(y) & L(y) \end{tikzcd}$$

If $x=y$ and $\varphi = id_x$, we have necessarily

$$L(\varphi) = (i_x)^\flat = id_{L(x)}$$ by uniqueness.

Similarly

$$\begin{tikzcd}
x \arrow[r, "i_x"] \arrow[d, "\varphi"'] & RL(x) \arrow[d, "RL(\varphi)"] \arrow[dd, bend left, "RL(\psi\circ\varphi)"] \\
y \arrow[r, "i_y"] \arrow[d, "\psi"'] & RL(y) \arrow[d, "RL(\psi)"] \\
z \arrow[r, "i_z"] & RL(z)
\end{tikzcd}
\qquad
\begin{tikzcd}
L(x) \arrow[d, "L(\varphi)"] \arrow[dd, bend left, "L(\psi\circ\varphi)"] \\
L(y) \arrow[d, "L(\psi)"] \\
L(y)
\end{tikzcd}$$

shows that $L(\psi\circ\varphi) = L(\psi)\circ L(\varphi)$. Hence L is a <u>functor</u>, $\mathcal{C} \xrightarrow{L} \mathcal{D}$.

For $L(a) \xrightarrow{\alpha} b$ we <u>define</u> $\alpha^\#$ as

$\alpha^\# = R(\alpha)\circ i_a$.

$a \xrightarrow{i_a} RL(a) \xrightarrow{R(\alpha)} R(b)$

Thus $(\alpha^\#)^\flat = \alpha$ by uniqueness

and $(\alpha^\flat)^\# = \alpha$ because $R(\alpha^\flat)\circ i_a = \alpha$.

Hence $\flat$ is a bijection, with inverse $\#$.

For $L(a) \xrightarrow{\varphi} b \xrightarrow{\beta} \tilde{b}$ we have

$$(\beta\circ\varphi)^\# = R(\beta\circ\varphi)\circ i_a = R(\beta)\circ R(\varphi)\circ i_a = R(\beta)\circ\varphi^\#$$

and for $\tilde{a} \xrightarrow{\alpha} a$ $\quad L(a) \xrightarrow{\psi} b$

$$(\psi\circ L(\alpha))^\# = R(\psi\circ L(\alpha))\circ i_{\tilde{a}} = R(\psi)\circ RL(\alpha)\circ i_{\tilde{a}}$$
$$= R(\psi)\circ i_a\circ\alpha = \psi^\#\circ\alpha$$

$\square$

<u>Remark</u> <u>Freyd's Adjoint Functor theorem</u> states the following. Let $\mathcal{C} \xleftarrow{R} \mathcal{D}$ be a continuous functor. Assume that $\mathcal{D}$ is complete and that $\mathcal{C}$ and $\mathcal{D}$ are locally small. Then R has a left adjoint $L : \mathcal{C} \to \mathcal{D}$ if and only if the <u>solution set condition</u> is satisfied:

(*) for every $a \in obj(\mathcal{C})$ there is a set (!) $I = I(a)$ and a family of morphisms $\alpha_i : a \to R(b_i)$, such that every morphism $a \xrightarrow{\alpha} R(b)$ factors as

$$\begin{tikzcd} a \arrow[rr, "\alpha"] \arrow[dr, "\alpha_i"'] & & R(b) \\ & R(b_i) \arrow[ur, "R(\beta_i)"'] & \end{tikzcd}$$

for <u>some</u> i and some $b_i \xrightarrow{\beta_i} b$.

The theorem can be used to show the <u>existence</u> of

$F : \underline{set} \to grp$ (Free) or $\beta : \underline{tych} \to \underline{comp}$ (Čech-Stone)

or $b : \underline{ltg} \to \underline{cgp}$ (Bohr)

It gives, however, no clue about the nature of these functors.