$\{y \mid f(y) \neq 0\}$; observe that y is not in the support of f if and only if y has a nbd on which f vanishes identically.

4.1 Definition Let Y be a Hausdorff space. A family $\{\kappa_\alpha \mid \alpha \in \mathscr{A}\}$ of continuous maps $\kappa_\alpha: Y \to I$ is called a partition of unity on Y if:

(1). The supports of the κ_α form a nbd-finite closed covering of Y.

(2). $\sum_\alpha \kappa_\alpha(y) = 1$ for each $y \in Y$ (this sum is well-defined because each y lies in the support of at most finitely many κ_α).

If $\{U_\beta \mid \beta \in \mathscr{B}\}$ is a given open covering of Y, we say that a partition $\{\kappa_\beta \mid \beta \in \mathscr{B}\}$ of unity is subordinated to $\{U_\beta\}$ if the support of each κ_β lies in the corresponding U_β.

Clearly, every space has a partition of unity subordinated to the covering by the single set itself.

4.2 Theorem Let Y be paracompact. Then for each open covering $\{U_\alpha \mid \alpha \in \mathscr{A}\}$ of Y there is a partition of unity subordinated to $\{U_\alpha\}$.

Proof: Shrink a precise nbd-finite refinement of $\{U_\alpha\}$ to get a nbd-finite open covering $\{V_\alpha\}$ with $\overline{V}_\alpha \subset U_\alpha$ for each α. Now shrink $\{V_\alpha\}$ to get a nbd-finite open covering $\{W_\alpha\}$ satisfying $\overline{W}_\alpha \subset V_\alpha$. For each $\alpha \in \mathscr{A}$, VII, **4.1**, gives a continuous $g_\alpha: Y \to I$, which is identically 1 on $\overline{W}_\alpha$ and vanishes on $\mathscr{C}V_\alpha$ (we take $g_\alpha \equiv 0$ if $V_\alpha = \varnothing$); each g_α has its support in U_α. Since $\{\overline{W}_\alpha\}$ is a nbd-finite covering, it follows that for each $y \in Y$ at least one, and at most finitely many, g_α are not zero, consequently $\sum_\alpha g_\alpha$ is a well-defined real-valued function on Y and is never zero. $\sum_\alpha g_\alpha$ is continuous on Y: every point has a nbd on which all but at most finitely many g_α vanish identically, so the continuity of $\sum g_\alpha$ on this nbd follows from that of each g_α, and by III, **9.4**, $\sum g_\alpha$ is therefore continuous on Y. The required partition of unity is given by the family of functions $\{\kappa_\alpha \mid \alpha \in \mathscr{A}\}$, where

$$\kappa_\alpha(y) = \frac{g_\alpha(y)}{\sum_\alpha g_\alpha(y)}.$$

We remark that in a normal space Y, the proof shows that a partition of unity subordinated to a given *nbd-finite* open cover exists; C. H. Dowker has shown that their existence for *each* open cover is equivalent to paracompactness of Y [*cf.* **5.5**(2)].

To give an application of **4.2**, note that if $\{\kappa_\alpha \mid \alpha \in \mathscr{A}\}$ is a partition of unity on Y, and if $\{\varphi_\alpha \mid \alpha \in \mathscr{A}\}$ is any family of continuous maps $\varphi_\alpha: Y \to E^1$, then the map $Y \to E^1$ given by $y \to \sum_\alpha \varphi_\alpha(y)\kappa_\alpha(y)$ is also continuous.

4.3 (C. H. Dowker) Let Y be paracompact. Assume that g is a lower, and G an upper, semicontinuous real-valued function on Y such that $G(y) < g(y)$ for each $y \in Y$. Then there exists a *continuous* $\varphi: Y \to E^1$ such that $G(y) < \varphi(y) < g(y)$ for each $y \in Y$.

Proof: For each rational r, let $U_r = \{y \mid G(y) < r\} \cap \{y \mid g(y) > r\}$; due to the semicontinuities, this is open; and because for each y there is some rational $\bar{r}$ with $G(y) < \bar{r} < g(y)$, the family $\{U_r\}$ is in fact an open covering of Y. Let $\{\kappa_r\}$ be a partition of unity subordinated to $\{U_r\}$; the required continuous function is $\varphi(y) = \sum_r r \cdot \kappa_r(y)$. For, let $y \in Y$ be given, and let $\kappa_{r_1}, \cdots, \kappa_{r_n}$ be all those functions whose support contains y; then $y \in U_{r_1} \cap \cdots \cap U_{r_n}$ so that $G(y) < r_i < g(y)$ for each $i = 1, \cdots, n$, and therefore

$$G(y) = G(y) \cdot \sum \kappa_{r_i}(y) < \sum r_i \kappa_{r_i}(y) = \varphi(y) < g(y) \cdot \sum \kappa_{r_i}(y) = g(y).$$

5. Complexes; Nerves of Coverings

The concept of a partition of unity subordinated to a given open covering has an alternative, more geometrical, interpretation. To develop this, we need two preliminary notions.

(1). Let $\mathscr{A}$ be any set. By an n-simplex σ^n in $\mathscr{A}$ is meant a set $(\alpha_0, \cdots, \alpha_n)$ of $n + 1$ distinct elements of $\mathscr{A}$; $\alpha_0, \cdots, \alpha_n$ are called the vertices of σ^n, and any $\sigma^q \subset \sigma^n$ is termed a q-face of σ^n.

5.1 Definition An abstract simplicial complex $\mathscr{K}$ over $\mathscr{A}$ is a set of simplexes in $\mathscr{A}$ with the property that each face of a $\sigma \in \mathscr{K}$ also belongs to $\mathscr{K}$.

With each abstract simplicial complex we will associate a standard topological space. For this we need

(2). Given $(n + 1)$ independent points $p_0, \cdots, p_n$ in an affine space, the open geometric n-simplex σ^n spanned by $p_0, \cdots, p_n$ is

$$\left\{\sum_0^n \lambda_i p_i \,\middle|\, \sum_0^n \lambda_i = 1, \quad 0 < \lambda_i \leqslant 1, \quad i = 0, \cdots, n\right\};$$

it is denoted by $(p_0, \cdots, p_n)$.

σ^n is the interior of the convex hull of $\{p_0, \cdots, p_n\}$ in the n-dimensional Euclidean space that these vertices span; for example, (p_0, p_1) is a segment without its end points, and (p_0, p_1, p_2) is a triangle without its boundary. The λ_i, $i = 0, \cdots, n$, are called the barycentric coordinates of

$$x = \sum_0^n \lambda_i p_i;$$

the closed geometric n-simplex $\bar{\sigma}^n = \overline{(p_0, \cdots, p_n)}$ consists of σ^n with its boundary, and is obtained by allowing $0 \leqslant \lambda_i \leqslant 1$ for $i = 0, \cdots, n$.

5.2 Definition Given any set $\mathscr{A}$, let $L(\mathscr{A})$ be a real vector space with finite topology, having a basis $\{b_\alpha\}$ in fixed 1-to-1 correspondence $b_\alpha : \alpha$ with the elements of $\mathscr{A}$, and let u_α be the unit point on the vector b_α. Given any complex $\mathscr{K}$ over $\mathscr{A}$, let $K \subset L(\mathscr{A})$ be the union of all open geometric simplexes $(u_{\alpha_0}, \cdots, u_{\alpha_n})$ for which $(\alpha_0, \cdots, \alpha_n)$ is a simplex in $\mathscr{K}$. The subspace $K \subset L(\mathscr{A})$ is called a polytope with vertex scheme $\mathscr{K}$ (or a standard geometrical realization of $\mathscr{K}$).

It is evident that the space K has the weak topology determined by the Euclidean topology on its closed simplexes, so that an $f: K \to Y$ is continuous if and only if it is so on each $\bar{\sigma}^n$. This implies that any two standard geometrical realizations K_1, K_2 of a given $\mathscr{K}$ are homeomorphic: for, to each $\sigma^n \in \mathscr{K}$ there correspond unique $\sigma_1^n = (p_0^1, \ldots, p_n^1)$ and $\sigma_2^n = (p_0^2, \cdots p_n^2)$ in K_1, K_2, respectively, and by barycentrically mapping each σ_1^n on the corresponding σ_2^n (that is $\sum_0^n \lambda_i p_i^1 \to \sum_0^n \lambda_i p_i^2$), the desired homeomorphism is obtained. Thus we can speak of *the* geometric realization of $\mathscr{K}$.

In a polytype, the star, St u_0, of a vertex u_0 is the set of all open geometric simplexes having u_0 as vertex. It is important to note that St u_0 is an open set in K: given any closed $\bar{\sigma} = \overline{(u_{\alpha_0}, \cdots, u_{\alpha_n})}$, its intersection with $K -$ St u_0 is either $\bar{\sigma}$ if no $u_{\alpha_i} = u_0$ or a face of σ if some $u_{\alpha_i} = u_0$; in either case, this intersection is closed in $\bar{\sigma}$, so $K -$ St u_0 is closed in K.

The process of associating with each open covering of a space a complex called its nerve is very important because it is one method for relating the topological to the algebraic properties of spaces; intuitively geometric realizations of nerves approximate the space with the finer covering giving the better approximation.

5.3 Definition Let $\{U_\alpha \mid \alpha \in \mathscr{A}\}$ be any covering of a space. Define a complex $\mathscr{N}$ over $\mathscr{A}$ by the following condition: $(\alpha_0, \cdots, \alpha_n)$ is a simplex of $\mathscr{N}$ if and only if $U_{\alpha_0} \cap \cdots \cap U_{\alpha_n} \neq \varnothing$. It is evident that $\mathscr{N}$ is indeed a complex, called the nerve of $\{U_\alpha \mid \alpha \in \mathscr{A}\}$. The standard geometric realization of $\mathscr{N}$ is called the geometric nerve of $\{U_\alpha \mid \alpha \in \mathscr{A}\}$ and is denoted by $N(U_\alpha)$.

The vertex of $N(U_\alpha)$ corresponding to the set U_α is denoted by u_α.

5.4 Theorem Let Y be any space and $\{U_\alpha \mid \alpha \in \mathscr{A}\}$ be an open covering. Then for each partition of unity subordinated to $\{U_\alpha\}$ there exists a continuous $\kappa: Y \to N(U_\alpha)$ such that $\kappa^{-1}(\text{St } u_\alpha) \subset U_\alpha$ for each α.

Proof: Let $\{\kappa_\alpha \mid \alpha \in \mathscr{A}\}$ be a partition of unity subordinated to $\{U_\alpha\}$, and define

$\kappa: Y \to N(U_\alpha)$ by $\kappa(y) = \sum_\alpha \kappa_\alpha(y) u_\alpha$. This is continuous: each $y \in Y$ has a nbd on which all but at most finitely many κ_α vanish, and since this nbd is mapped into a finite-dimensional flat in $L(\mathscr{A})$, the addition is continuous (*cf.* Appendix I, **4**), so κ is continuous on that nbd and its continuity on Y results from III, **8.3**. Since $\sum_\alpha \kappa_\alpha(y) = 1$, $\kappa(y)$ is in fact a point of the closed geometric simplex spanned by $\{u_\alpha \mid \kappa_\alpha(y) \neq 0\}$. The inverse image of St u_{α_0} consists of all y for which $\kappa_{\alpha_0}(y) \neq 0$, and because the support of κ_{α_0} is in U_{α_0}, we have

$$\kappa^{-1}(\text{St } u_{\alpha_0}) \subset U_{\alpha_0}$$

as required.

It should be observed that if $V \subset Y$ is an open set intersecting the supports of only the finitely many $\kappa_{\alpha_1}, \cdots, \kappa_{\alpha_n}$, then $\kappa(V) \subset \overline{(u_{\alpha_0}, \cdots, u_{\alpha_n})}$.

5.5 Remark It is known (*cf.* Appendix I, **5.2**) that the geometric nerve $N(U_\alpha)$ is always a paracompact space. Using this fact, we can prove

(1). A continuous $\kappa: Y \to N(U_\alpha)$ satisfying $\kappa^{-1}(\text{St } u_\alpha) \subset U_\alpha$ for each α exists if and only if there is a partition of unity subordinated to $\{U_\alpha\}$.

The "if" is **5.4**; the "only if" follows by finding a partition of unity $\{\kappa_\alpha \mid \alpha \in \mathscr{A}\}$ subordinated to the open cover $\{\text{St } u_\alpha \mid \alpha \in \mathscr{A}\}$ of the paracompact $N(U_\alpha)$ and defining $\lambda_\alpha: Y \to I$ by $\lambda_\alpha = \kappa_\alpha \circ \kappa$.

(2). Y is paracompact if and only if for each open covering $\{U_\alpha\}$ there is a subordinated partition of unity.

The "only if" is **4.2**; the "if" follows by finding a nbd-finite refinement $\{V_\beta \mid \beta \in \mathscr{B}\}$ of the open covering $\{\text{St } u_\alpha \mid \alpha \in \mathscr{A}\}$ in $N(U_\alpha)$; then

$$\{\kappa^{-1}(V_\beta) \mid \beta \in \mathscr{B}\}$$

is the desired nbd-finite refinement of $\{U_\alpha\}$. There is a simpler proof of "if" which uses the geometry, rather than the paracompactness, of $N(U_\alpha)$: letting N' be the barycentric subdivision of $N(U_\alpha)$ [*cf.* Appendix I, **5**] and using stars in N', we have that $\{\kappa^{-1}(\text{St } p') \mid p' \text{ a vertex of } N'\}$ is a barycentric refinement of $\{U_\alpha \mid \alpha \in \mathscr{A}\}$. This indicates the origin of the term *barycentric refinement*.

6. Second-countable Spaces; Lindelöf Spaces

In this section, we study two properties of spaces related to the behavior of their open coverings; it turns out that when any one of them is present, weak separation properties become very strong.

6.1 Definition A Hausdorff space is 2° countable (or, satisfies the second axiom of countability) if it has a countable basis.

In recent literature, the least cardinal of a basis for a space X is called the weight of X; thus, X is 2° countable if it has weight $\leqslant \aleph_0$.

Ex. 1 E^n is 2° countable, as seen in III, **2**, Ex. 3. A countable discrete space is