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SS 2012

1.Übung zur Vorlesung Gebäude

Please hand in your solutions on the morning of Friday 13 April before the lecture.

Aufgabe 1.1 (1. Projective Geometry)

Let $\Delta(K^{n+1})$ be the flag complex of the vector space K^{n+1} over the field K. Prove that

(a) each flag is contained in a maximal flag
(b) each maximal flag is generated by an ordered n-tuple of n linearly independent vectors
(c) $GL(n+1, K)$ acts transitively on the maximal flags
(d) Consider the maximal flag $span\{e_1\} \subset span\{e_1, e_2\} \subset \ldots \subset span\{e_1, e_2, \ldots, e_n\}$ where $e_1, e_2, \ldots e_{n+1}$ is the standard basis for K^{n+1}. Describe the stabiliser of this flag in $GL(n+1, K)$.
(e) Prove that the maximal flags are in one-to-one correspondence with the conjugates of this group in $GL(n+1, K)$.

Aufgabe 1.2 (2. Apartments)

Given a basis B of K^{n+1}, the subcomplex $\Sigma(B)$ of $\Delta(K, n)$ is an apartment. Show that

(a) the apartments are in one-to-one correspondence with the frames, where a frame is a set of 1-dimensional subspaces of K^{n+1} of cardinality $n+1$ which together span all of K^{n+1}.
(b) Show that $SL(n+1, K)$ acts transitively on the apartments.
(c) Describe the subgroup of $SL(n+1, K)$ which fixes every chamber of the apartment $\Sigma(B)$ where $B = \{e_1, e_2, \ldots e_{n+1}\}$, the standard basis.
(d) Show that each apartment is isomorphic to the barycentric subdivision of the simplicial complex consisting of the $n-1$-dimensional faces of an n-simplex. (Hint: the standard n-simplex is the convex hull of the vectors $b_1, \ldots b_{n+1}$ in $\mathbb{R}^{n+1}$. Find a one-to-one correspondence between the maximal simplices in the barycentric subdivision and the set of total orderings of the set $\{b_1, \ldots, b_{n+1}\}$.

Aufgabe 1.3 (3. Projective Planes)

An incidence structure (P, L, I) is a set of points P, a set of lines L and an incidence relation $I \subset P \times L$. Two elements $p \in P$ and $l \in L$ are said to be incident if $(p, l) \in I$. An incidence structure is a projective plane, if and only if:

(i) Given two distinct points, there is exactly one line incident to both of them.
(ii) Given two distinct lines, there is exactly one point incident to both of them.
(iii) There are at least four points, such that no line is incident to any three of them. (non-degeneracy)

(a) Let K be a field. Show that $PG(K, 2)$ forms a projective plane where we take the 1-dimensional subspaces of K^3 as the points, the 2-dimensional subspaces as the lines, and say that a point and line are incident if the former is a subspace of the latter. These are the classical projective planes.

(b) Let F_q be a finite field of order q. Show that in $PG(F_q, 2)$ there are exactly $q + 1$ points on every line and $q + 1$ lines passing through every point, and that there are $q^2 + q + 1$ points altogether.
(*) A projective plane is said to be Desarguesian if the following is true. Given six points A, B, C, a, b, c, such that the lines Aa, Bb, Cc intersect at one point, the intersections of AC and ac, AB and ab, BC and bc, are collinear. Prove that the classical projective planes are Desarguesian. (Hint: You may want to make use of the fact that $PG(K, 2)$ can be embedded in $PG(K, 3)$.) There exist finite non-Desarguesian projective planes, but the smallest have 91 points and 91 lines.