Prof. Dr. L. Kramer
Dr. Rupert McCallum
Antoine Beljean

SS 2012

9.Übung zur Vorlesung Gebäude

Please hand in your solutions on the morning of Friday 15 June before the lecture.

Aufgabe 9.1 (1. The Tits System of $\mathrm{GL}_{m+1}(K)$)

Let K be a field. Let $G = \mathrm{GL}_{m+1}(K)$ and let B be the subgroup of upper triangular matrices with respect to a fixed basis, and let N be the subgroup of monomial matrices (matrices with exactly one entry in each column and row different from 0), and let T be the subgroup of diagonal matrices. Denote by I_j the identity matrix of size j. Then for $1 \le i \le m$ let

$$s_i = \begin{pmatrix} I_{i-1} & & & \\ & 0 & 1 & \\ & -1 & 0 & \\ & & & I_{m-i} \end{pmatrix}, G_i = \left\{ \begin{pmatrix} I_{i-1} & & \\ & A & \\ & & I_{m-i} \end{pmatrix} : A \in \mathrm{GL}_2(K) \right\}$$

Let $S = \{s_i T : 1 \le i \le m\} \subset N/T$. Then (G, B, N, S) is a Tits system. The proof in the lecture used the following two steps - prove these:
(a) Write $B^n = n^{-1} B n$ for an element $n \in N$. Let $w = nT \in W \cong N/T \cong \mathrm{Sym}(m+1)$. Then show for all $1 \le i \le m$:

$$B^n \cap G_i = \left\{ \begin{array}{ll} \left\{ \begin{pmatrix} I_{i-1} & & & \\ & * & * & \\ & 0 & * & \\ & & & I_{m-i} \end{pmatrix} \right\} & w(i) < w(i+1) \\ \left\{ \begin{pmatrix} I_{i-1} & & & \\ & * & 0 & \\ & * & * & \\ & & & I_{m-i} \end{pmatrix} \right\} & w(i) > w(i+1) \end{array} \right\}$$

(b) Conclude that $G_i = (B \cap G_i)(B^n \cap G_i) \cup (B \cap G_i) s_i (B^n \cap G_i)$.

Aufgabe 9.2 (2. Parabolic Subgroups)

Let (G, B, N, S) be a Tits System.

(a) Show that for all $w \in W$ we have $\langle BwB \rangle = \langle B \cup wBw^{-1} \rangle$.
(b) For all $J \subset I$ let $W_J = \langle J \rangle$ and $P_J = BW_J B$. Show that each parabolic subgroup P_J is self-normalising, i.e. $N_G(P_J) = P_J$. Also show that no two distinct parabolic subgroups are conjugate.