Prof. Dr. L. Kramer
Dr. Rupert McCallum
Antoine Beljean

SS 2012

5.Übung zur Vorlesung Gebäude

Please hand in your solutions on the morning of Friday 11 May before the lecture.

Aufgabe 5.1 (2. Joins)

(2 marks) Suppose that X, Y are two simplicial complexes.

(a) Show that the join $X * Y$ is also a simplicial complex. Describe the join if Y consists of a single vertex and show that in this case $|X * Y|$ is contractible.
(b) Show that if X and Y are chamber complexes, then $X * Y$ is also a chamber complex. Describe the chambers and the chamber graph of $X * Y$.

Aufgabe 5.2 (3. Type-preserving automorphisms)

(4 marks) Suppose that (W, I) is a Coxeter system, and let $\Sigma = \Sigma(W, I)$. Recall the type function

$$t : \Sigma(W, I) \to 2^I,\ wW_J \mapsto I \setminus J.$$

Show that W is the group of all automorphisms of Σ over I. (Hint: Prove that an automorphism over I which fixes a chamber is the identity.)

Aufgabe 5.3 (4. The finite symmetric groups are Coxeter complexes)

The Coxeter group of type A_n is defined to be the Coxeter group with generators $i_1, \ldots, i_n$ and Coxeter diagram

Thus i_j and i_k commute if $|j - k| \geq 2$ and $i_k i_{k+1}$ has order 3, for $1 \leq k \leq n - 1$.

(a) (2 marks) Consider the symmetric group $Sym(n+1)$ of permutations of the set $\{1, \ldots n+1\}$. If W is a Coxeter group of type A_n, show that there exists a surjective homomorphism $W \to Sym(n + 1)$ which maps i_k to the transposition $(k, k + 1)$ for all $k \in \{1, 2, \ldots, n\}$.
(b) (2 marks) Let $J := \{i_1, i_2, \ldots i_{n-1}\}$. Show (for example by induction) that the cosets

$$W_J,\ i_n W_J,\ i_{n-1} i_n W_J, \ldots,\ i_1 \ldots i_n W_J$$

exhaust all of W and hence that W has cardinality at most $(n + 1)!$. Conclude that the group W is isomorphic to $Sym(n + 1)$.