Mathematical Physics

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Clay Mathematics Institute Millennium Prize [10^6\$] Problem

5. Yang-Mills Existence and Mass Gap

Prove that for any compact simple gauge group *G*, a non-trivial quantum Yang-Mills theory exists on $\mathbb{R}^4$ and has a mass gap $\Delta > 0$. Existence includes establishing axiomatic properties at least as strong as those of [Wightman, Osterwalder-Schrader].

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There is not even a toy model of a 4D QFT!

Quantum geometry

Argument due to Wheeler (1950s), made precise by Doplicher-Fredenhagen-Roberts (1995):

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Our standard QFT violates this principle (admits arbitrarily small distances).

Need to develop QFT on quantum geometries.

First success stories

Take toy quantum geometry which admits computation.

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Theorem (Disertori-Gurau-Magnen-Rivasseau, 2006)

Assume Planck volume = volume of universe.
Then the β-function is zero to all orders in perturbation theory.

This is a precious result!

Immediate question: Can we construct the model?

Inside an atom of geometry

- Take toy quantum geometry
- Adjust V = volume of universe = volume of Planck cell

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1. The limit $V \to \infty$ of the 4D scalar Euclidean QFT is exactly solvable for any coupling constant $> -\frac{1}{\pi}$.
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3. Diagonal correlation functions have interpretation as Schwinger functions in position space.
   - They have full Euclidean symmetry (of standard, not quantum!) 4D space.
   - They are blind to colour (confinement / darkness).

Work in progress: 1 Time 2 Quantum gravity

1. QFT on space-time arises if Schwinger functions are reflection-positive.
   - Overwhelming numerical evidence and partial analytic proof that 2-point function is reflection positive.
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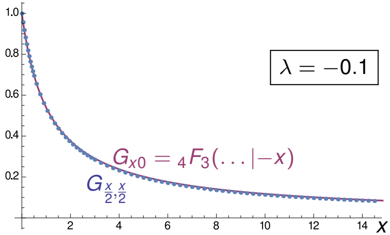
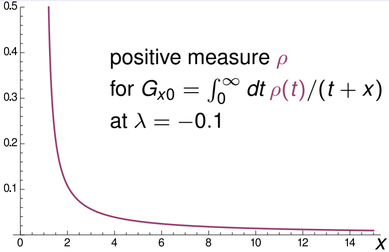
2. This subject has inspired enormous progress with combinatorial quantum gravity.
   Breakthrough by Razvan Gurau (2011), many followers
    - Jins de Jong [SFB 878]: construction of a quartic analogue of the Kontsevich model
    - Carlos Pérez-Sánchez [DAAD]: Schwinger-Dyson equations in coloured tensor models

Supplement: Reflection positivity





- Right: positive measure ρ supported on $]1, \infty[$
- Left: red curve is auxiliary function $G_{x0} = \int_0^\infty \frac{dt\, \rho(t)}{t+x}$ which solves fixed point problem up to 10^{-8}

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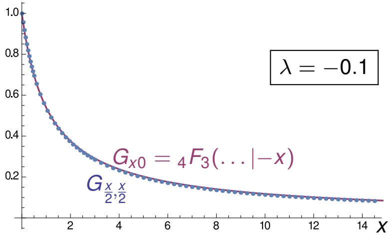

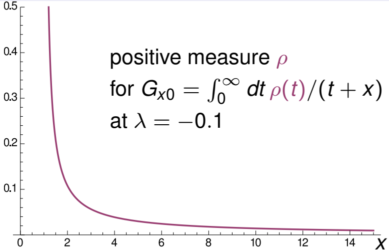

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- Left: red curve is auxiliary function $G_{x0} = \int_0^\infty \frac{dt\,\rho(t)}{t+x}$ which solves fixed point problem up to 10^{-8}
- Defines diagonal function $G_{\frac{x}{2}\frac{x}{2}}$ (Schwinger function).
- Reflection positivity is existence on a blue positive function ϱ on the right (the Källen-Lehmann mass spectrum) such that $G_{\frac{x}{2}\frac{x}{2}} = \int_0^\infty \frac{dt\,\varrho(t)}{t+\frac{x}{2}}$