Structured Chaos[1]

Approach to a subject of modern physics by means, of simple systems

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Die Natur (kann) auch *selbst im* Chaos nicht anders,
als regelmäßig und ordentlich *verfahren.*
Kant

Introduction

Chaotic behaviour of physical systems is not new. A gaz for example in a chaotic system par excellence. The root of the word gaz in chaos. However, chaos does not only arise in many particle systems. For a long time it is known that systems with only a few degrees of freedom may be chaotic. About one hundred years ago, Henri Poincaré criticized the mechanical worldview of classical physics. Although he directed his attack against the bastion of Newtonian physics, celestial mechanics, until very recently he and his followers have not been heard. In the meantime, a drastic change in attitude seems to take place: Chaotic phenomena are not only detected in all established areas of natural science but also are taken for serious, and investigated. Of course, it cannot yet be deckled whether this is a change in paradigm in the sense of T.S. Kuhn [1], although there are many hints for it. Especially, the change of the physical method which may be characterized by the slogan: it In. better to determine a reliable, general frame for the future systems behaviour than to make exact but incorrect quantitative predictions.

In the following we shall sketch the main features of such a qualitative procedure at the concrete example of a rotating pendulum.

What is a chaotic system?

We cannot investigate here the epistemological problem why chaos in invading our scientific world just now. Only the following points shall be mentioned:

- Many problems of the modern world show that the fundamentum especially of physics relying on deterministic, and reversible laws in too small.

- The rapid development of powerful and simply to handle computers have made it possible to solve nonlinear differential equations numerically without great effort making linearizations more and more obsolete.

- By taking into account nonlinear relations the attention has, finally, been directed on those apparently unaccessible phenomena of which the chaotic ones are the most spectacular and which have been overlooked and repressed until very recently.

A chaotically behaving system in described by nonlinear differential equations. Of Course, nonlinear differential equations have unique solutions. In thin respect, chaos In quite a deterministic phenomenon. But practically, it in indeterministic in that it exhibits a nonpredictable behaviour.

Something very similar in known from a die: What In more deterministic than the throw of a die? it in completely described by simple mechanical laws. If somebody would throw a die two times in the same way, it would show in either case the same result. But nobody and no machine are able to throw a die exactly in the same way. Therefore, the thrown number in not the result of skill but of mere chance.

The reason for thin behaviour in, that the die shows a sensitive *dependence on* its initial conditions. As it depends on small differences in the initial conditions whether the die, just before it comes to rest, will still be able to overroll an edge or *whether it* will fall back. The existence of such "decision points" by which neighboring paths (due to almost

[1] In: H. Kühnelt (Ed.): Interdisciplinary Aspects of Physics Education. Singapore: World Scientific 1990, pp. 283 - 300

equal starting points) are scattered apart in responsible for thin sensitivity.

A chaotic system has such decision points all along the path (orbit). In order that In such a system chaotic phenomena will actually occur the values of the external parameters and of the initial conditions must permit that the decision points are within the range of the orbits. Periodically driven overturning pendula or the rotating pendulum described below represent examples of a class of chaotic systems the upper unstable equilibrium point of which In such a decision point. Starting with appropriate values of the parameters and initial conditions these points "determining" the further course of the orbit are attained repeatedly. Therefore, any prediction in practically impossible.

Back to geometry

Due to their sensitive dependence on the initial conditions, since very recently, chaotic systems were regarded as being unstable, and therefore physically unaccessible. But only a stable system which can remove small disturbances by itself may be investigated with regard to regularities.

A remarkable progress has been achieved by the topologist ***Smale***. He did no longer examine the local behaviour of a system but strived for a more global understanding: Instead of persuing single orbits sequentially he tried to comprehend the ensemble of all possible orbits simultaneously. By doing so, he finally succeeded to describe chaos and instability as totally different kinds of behaviour: A locally unpredictable *s*ystem could be quite stable from a global point of view. The hope was that globally stable systems showing roughly the same pattern *of* behaviour should be geometrically similar in their outlines.

In persuing thin idea a transition to geometry has been performed which was one of the most important steps towards chaos research: A kind of qualitative investigation has replaced the usual quantitative analysis. Therefore, beautiful pictures are a typical result of chaotic dynamics. (see. e.g. our beautiful “Julia” preceeding thin text who represents the basins of two attractors of the rotating pendulum investigated below. Only eyes, mouth, and nose did not arise from calculation but from our imagination.)

Geometrical methods are not only of a direct illustrative power. Moreover, they permit to exploit, for the use in scientific research, the creative possibilities of the human ability to recognize patterns: The detection of regularities within the geometrical figures representing the chaotic dynamics should allow to get insight into the possible behaviour of the system.

Phase space*:* Those geometric figures may be got by representing the systems behaviour by orbits in the phase space. The phase space in spanned by a set of systems parameters such that each possible state of the system in a point in this space. The temporal development of the system in then described by point series, so called orbits in space.

Dissipative structures**:** In the following, we shall confine to so called dissipative systems, which are thermodynamically open, flown through by energy and sometimes matter, They are of great importance for the modelling of real Systems. Such dissipative Systems tend, in course of time, towards a stationary final state adopting a characteric structure (dissipative structure). In thin stationary state, the System absorbs the same amount of high value energy (resp. matter) as it looses by dissipation so that the energy within the system stays constant. Therefore, not the energy itself but the dissipation of energy in responsible for the creation and the maintenance of the structure [2].

Example: Rotating pendulum

In the following, we shall confine to dissipative Systems with a periodic drive where the energy in supplied at a constant frequency. Especially, we shall demonstrate our results by means of the forced vibrations of a rotating pendulum (in Germany known as Pohlsches Rad) which has been made excentric by a small additional mass. As the experiments with such a system have already been described elsewhere [e.g. 3] we shall present the results of numerical simulations which show the rather manifold dynamics of the System. The following parameters of the rotating pendulum (fig. 1) may easily be varied experimentally:

- the additional mass m,
- the frequency of the drive Ω/Ω_0 (in units of the eigenfrequency Ω_0.of the corresponding harmonic pendulum),
- the amplitude of the drive f, which has been normalized on values between 1 and 2,
- the damping current I,
- the starting angle φ_0,
- the starting angular velocity $\dot{\varphi}_0$
- the central position of the drive α.

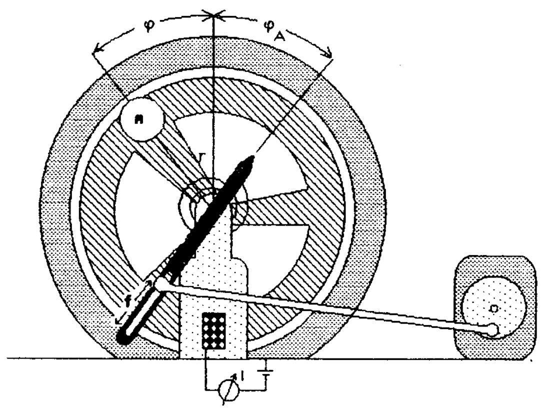

Fig. 1: The rotating pendulum with additional mass m

Equation of motion: The equation of motion of the pendulum is derived from the angular momenta excerted by the spiral Spring and the additional mass on the wheel. In addition, the spring in harmonically modulated by the drive:

$$\theta\ddot{\varphi} + \beta\dot{\varphi} = -D\left(\varphi - \varphi_A\right) + mgr\sin\varphi$$

With

$$\varphi_A = \alpha + \alpha_1\cos\Omega t,\ \ \rho = \frac{\beta}{\theta},\ \ \Omega_0^2 = \frac{D}{\theta},$$

$$r_0 = \frac{mgr}{\theta},\ \ F = \frac{D\alpha_1}{\theta}$$

it follows

$$\ddot{\varphi} = -\rho\dot{\varphi} - \Omega_0^2\varphi + \Omega_0^2\alpha + r_0\sin\varphi + F\cos\Omega t\,.$$

Thin equation in integrated numerically by a Runge-Kutta-procedure of 4th order. In most cases a constant integration step width of 50 steps/driving period turned out to be sufficient. The potential due to the spring and the additional mass within which the pendulum in "moving" reads as follows:

$$U\left(\varphi\right) = \frac{1}{2}\Omega_0^2\varphi^2 - \Omega_0^2\alpha\varphi + r_0\left(\cos\varphi - 1\right) \qquad (*)$$

In order to be able to compare the results of the simulation with the experiment we have Inserted the measured values of our pendulum into the equation of motion. and especially, determined experimentally the dependence of the damping on the current. We have restricted our investigations to the influence of the damping current on the Systems behaviour, thus keeping the following parameters at a constant value:

$$m = 25g,\ \ \frac{\Omega}{\Omega} = 0.5,\ f = 1.7 \text{ und } \alpha = 0 \qquad (**)$$

Attractors

The phase space of the rotating pendulum is spanned by the displacement angle φ, the angular velocity $\dot{\varphi}$, and the phase of the drive (resp. the time). Considering the development of the system within thin space the orbits of a regular final behaviour spiral around the time axis. The complexity of the orbits may be reduced by projecting them to a plane perpendicular to the time axis. Then, in the simplest case, the spiral runs back Into itself after each period, thus forming a *closed loop In* the *two* dimensional phase Space (fig. 2, centre) called limit cycle. Generally, such a figure "attracting" the final behaviour of a system in called **attractor.**

Another means of reducing complexity can be achieved by taking Into account the periodicity of the drive: By bending the time axis back to the origin, all the values of the phase of the drive φ_A modulo 2% are identified: The orbits *then s*piral around a *Gugelhupf like* torus leaving *the phase* space at $\varphi_A = 2\pi$, and, at the same time, reentering ft at $\varphi_A = 0$.

Poincaré section: In general, e.g. in the case of very complex behaviour of the System, such torus still contains too much Information. Therefore, Within the realm of chaotic dynamics it has become convenient to reduce complexity further by cutting the "Gugelhupf" into slices, thus, recording only the penetration points of the orbits (in the case of the simple limit cycle there in only one point (fig. 2, right». Each slice, called Poincaré section. corresponds to a *flashlight p*icture *of* the systems behavior taken at a certain *value of* the phase *of* the drive. Thus, taking a Poincaré section means considering the system stroboscopically at the rhythm of the drive.

Strange attractor. If a dissipative system does not only show regular but also irregular (chaotic) behaviour it in instructive to observe to what kind of final behaviour the system in tending in such a case. Different from what one might expect, namely that the Systems behaviour would be unstable and stochastic, even In thin case, it may be represented by a characteristic attractor. However, thin attractor in somewhat strange: Although the orbits are confined to a certain area in phase space as In the regular case they never meet again. They gradually fill out a compact area in phase space. Therefore, thin chaotic kind of an attractor is called strange attractor (fig. 2e). By means of such a geometric figure the analytically undeterminable may at least be visualized. In thin respect chaos appears not quite so chaotic in the sense of unpredictable as has been presumed until very recently. Thin may be one of the reasons for the interest in chaotic phenomena:

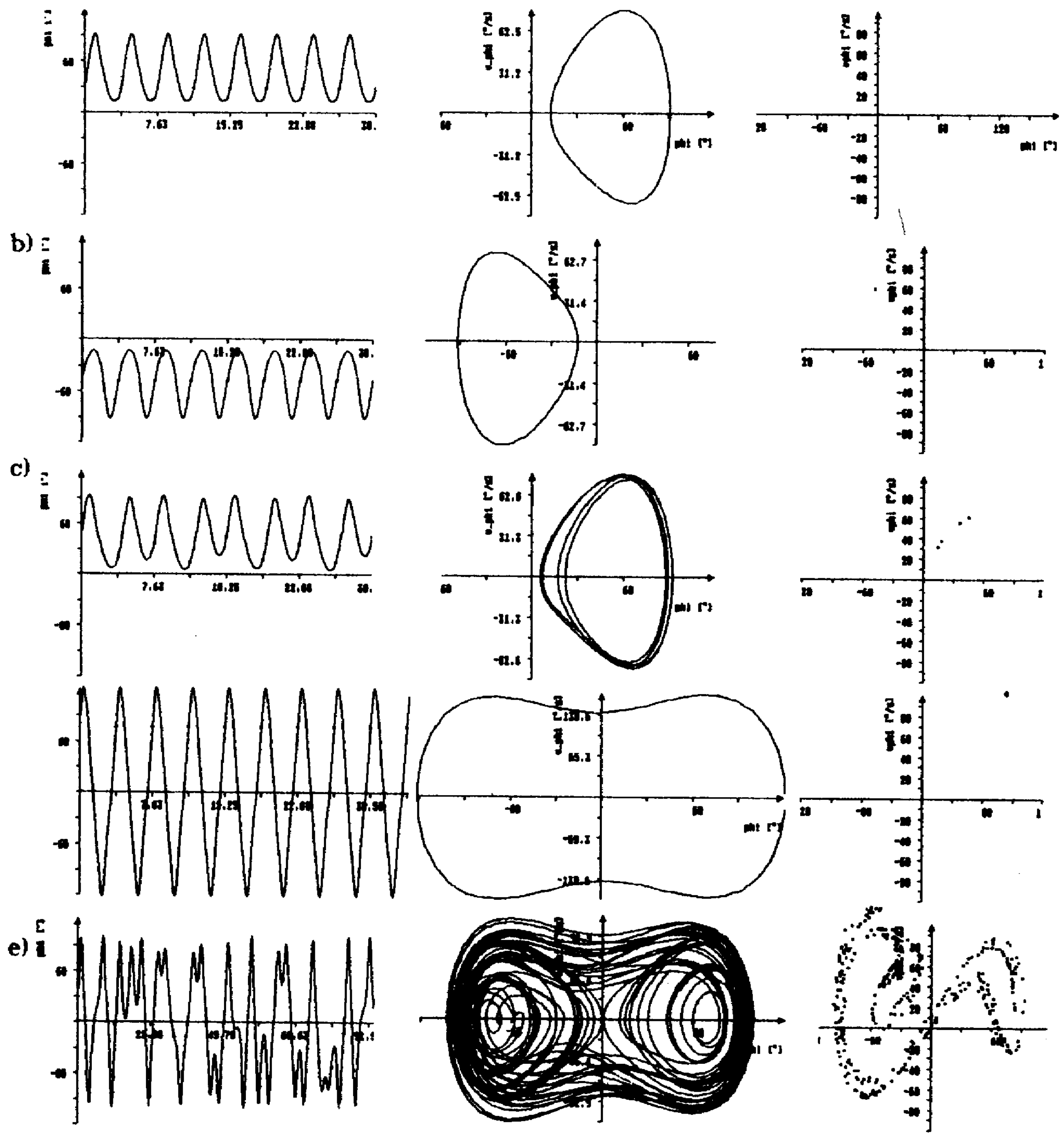

Fig. 2: Fig. 2: Limit cycle attractors (a-d) of regular vibration modes and strange attractor (e) of a chaotic vibration in different representations: ν(t)- diagram (left), phase diagram (centre), and Poincaré section (right). The vibration modes have been recorded for different damping currents, all other parameters are the same: a) and b) I = 528 mA, c) I = 515 mA, d) I = 250 mA, and e) I = 300 mA. a) and b) differ in the starting angle: a) $\nu_0 = 0$. b) $\nu_0 = -30°$

Although the single orbits cannot be predicted the behaviour of all orbits in represented by a characteristic object in phase space the structure of whi~2h may be subjected to detailed investigations.

In order to give at least an indication of what might be the outcome of chaos research, let us again use the metaphor of the die. Like a chaotic system the die behaves, globally considered, totally stable: Every throw results in one of the numbers 1 to 6. Moreover, these numbers appear with the same probability of 1/6. These facts have been the basin of a well established mathematical discipline, the theory of probability, which has become fundamental for modern science. There might be a similar de-

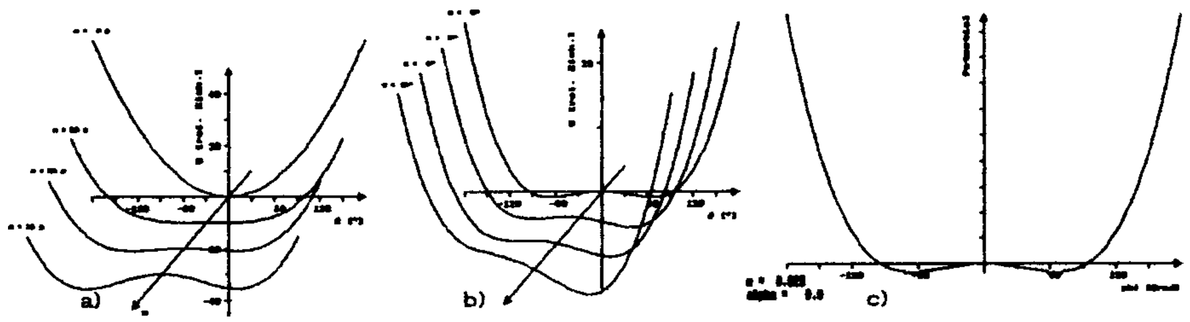

Fig. 3: Fig. 3: The potential of the rotating pendulum with additional mass a) for different masses m (0, 20, 25, 35 g), b) for different central positions of the drive α (0, 3, 6, 15°). c) for m = 25 g and α =0°.

velopment in chaos research, although, at present. there cannot be made any prediction.

Potential

The shape of the potential U(*) depends on m and a . Fig. 3a shows the change of U for increasing m and α = 0, fig. 3b the change for increasing α and m = 25 g. Increasing mass m leads to a continuous phase transition between different equilibrium positions whereas the variation of ix effects a discontinuous phase transition (see 141, [51), which easily may be demonstrated experimentally.

Fig. 3c shows the potential for the parameter values used in the present simulation. The limiting cases of regular behaviour may be recognized by inspecting the curve: the pendulum In swinging nearly harmonically if the damping in very great or very small. In thin cases the amplitude in too small or to large to take notice of the finite potential barrier at φ = 0. The anharmonicity of the potential at large damping currents makes itself felt by the fact that the pendulum may oscillate around two equilibrium positions (see fig. 2a and b). Which kind of final behaviour will actually occur depends on the initial conditions.

The deviation from the harmonic behaviour in getting more prominent if the pendulum oscillates differently on both sides of the potential barrier and also exhibits different oscillation modes within each hollow according to the initial conditions: As an example fig. 4 shows the phase portraits of three coexisting stationary modes at 1 = 514.25 mA. it should be pointed out that, in thin case, the orbits in phase space are closed only after several turns, i. e. after several cycles of the drive (fig. 4a: 4-cycle; fig. 4b: 3-cycle; fig. 4c: 16-cycle). They represent different stages of a period multiplication sequence (see below).

The all deciding beginning

Basins of coexisting attractors: What kind of final behaviour will actually be adopted depends on the choice of the initial conditions. Complete information about thin dependence may be obtained by systematically varying the initial conditions and investigating which final behaviour will result accordingly: Fig. 5 shows two examples of the basins of coexisting oscillation modes with the period of the drive (1-cycles).The diagrams are created by starting the system for each point with the corresponding initial conditions and coloring the point according to the resulting final behaviour. In some regions of the figures, a kind of fractal layered structure of the borders may be recognized when looking at it by increasing magnification.

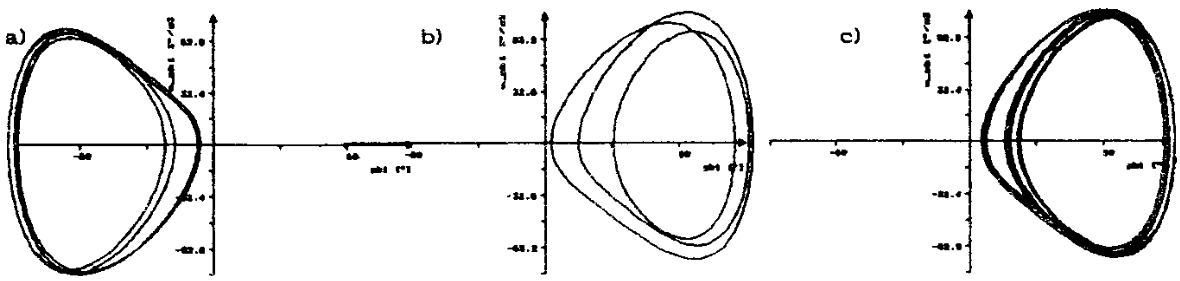

Fig. 4: Coexisting stationary vibrations at I = 514.25 mA: a) φ_0 = -30°, b) $\varphi_0 = 0^0$ and c) φ_0 = 10°.

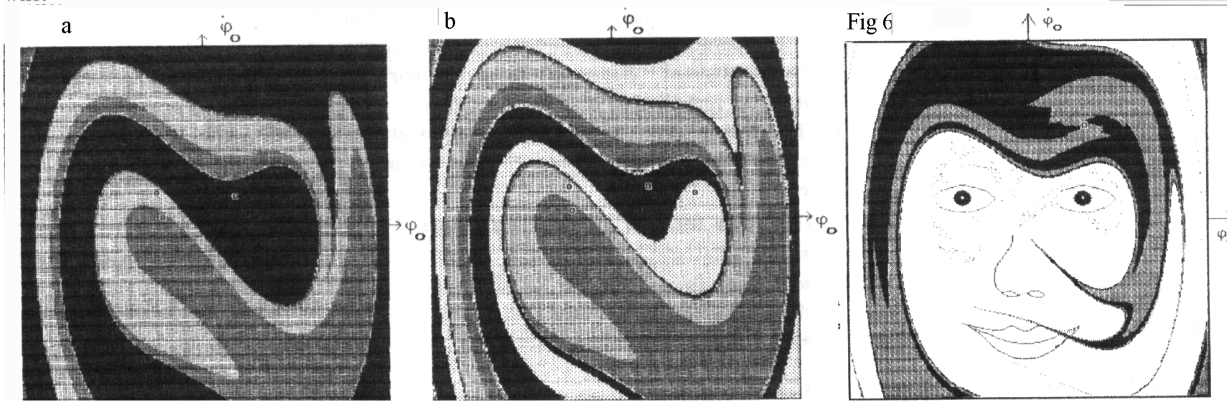

Fig. 5: Basins of different regular attractors: In the plane of initial conditions all points are colored according to the final behaviour by which they are "attracted": a) I = 535 mA, b) I = 540 mA. ($-180° < \varphi_0 < 180°$; $\frac{-300°}{s} < \dot{\varphi}_0 < \frac{300°}{s}$)

Fig. 6: Coexisting basins of a regular (dark) and a chaotic (white) attractor (-180° < φ < 180°; -300°/s < φ· < 300°/s)

At the transition from I = 535 mA to I = 540 mA the basin of the attractor (at $\varphi = 31°$; $\dot{\varphi} = \frac{51°}{s}$, dark) splits into two basins: A unstable final state has become stable. Even chaotic vibration modes may coexist with regular ones: Our "Julia" (fig. 6) in nothing else but a drawing of the basins of a regular (dark) and a chaotic attractor, which contains apart from some artistic changes the following informations:

- The Poincaré section of the chaotic attractor has been inserted into the basin of the attractor (its shape in known from fig. 2e).
- The points of the other basin have been given two gray colors according to their different "distances" from the corresponding attractor. The attractor itself is represented a by small square.

The name "Julia" reminds the mathematician G. Julia who invented the so called Julia-set, representing e.g. the boundary between the two basins (see e.g.[6]).

Transition between different attractors: The ex-

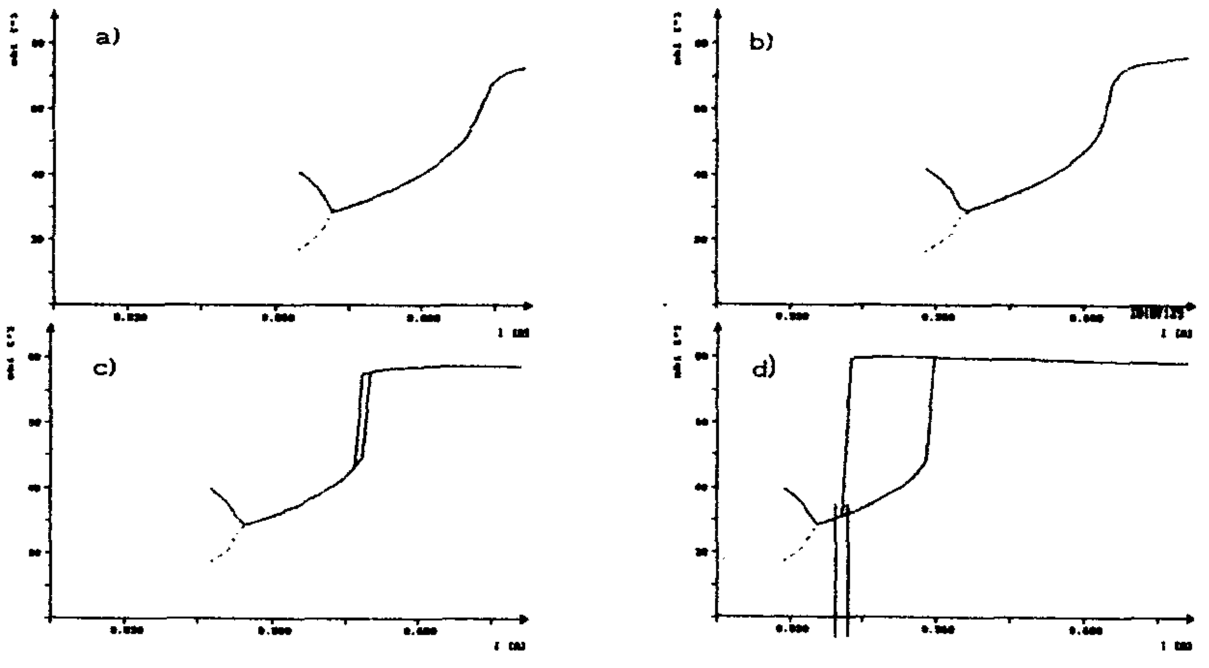

Fig. 7: Transition between two different final states. The displacement angle at the driving phase 0 has been recorded. The diagramms have been generated by first steadily increasing the damping current from low to high values and than back from high to low values during the stationary oscillation. a) f = 2.0, b) f = 1.95, c) f= 1.85. d) f = 1.70. In d) the values corresponding to fig. 5a) and b) have been indicated.

istence of unstable and metastable final states indicates a discontinuous transition between two different modes of final behaviour of the system if the damping current in changed during the oscillation such that after each small change transients can die away. The corresponding transitions have been depicted in fig. 7.

Temporal development of the orbits: We already mentioned that, globally regarded, chaotic behaviour is as stable as regular behaviour, a reminiscence of the underlying deterministic system. However, pursuing the course of single orbits there In a totally different behaviour: While in the chaotic case very small differences in the initial conditions are exponentially increasing to arbitrarily large differences in the regular case even large differences will be eliminated in course of time: The orbits "forget" their different origins and approach eachother until they are undistinguishable. Even in the chaotic case the initial conditions are forgotten in that the orbits, finally, will "fill up" all the area of the chaotic attractor. But in difference to the regular case the neighbourly relations only exist for a short time. After that, every neighborhood between orbits no matter how close it was gets lost as it manifests itself in an impressive way by the total mixing of the orbits. We shall demonstrate these facts at our rotating pendulum by following the fate of the orbits for 1000 starting points which - in the regular case - are distributed over an interval of 10 degres, and - in the chaotic case - are concentrated at an interval of 1 degree (fig. 8). Regarded in the rhythm of the periodic drive at equivalent values of the phase the basically different behaviour manifests itself already after a few periods. While, in the regular ease, the interval of the starting points in shrinking more and more to a point (representing a limit cycle), in the chaotic case, the points of the very small starting interval are drifting apart and, all of a sudden, spread over all the area of the chaotic attractor (sensitivity according to the initial conditions).

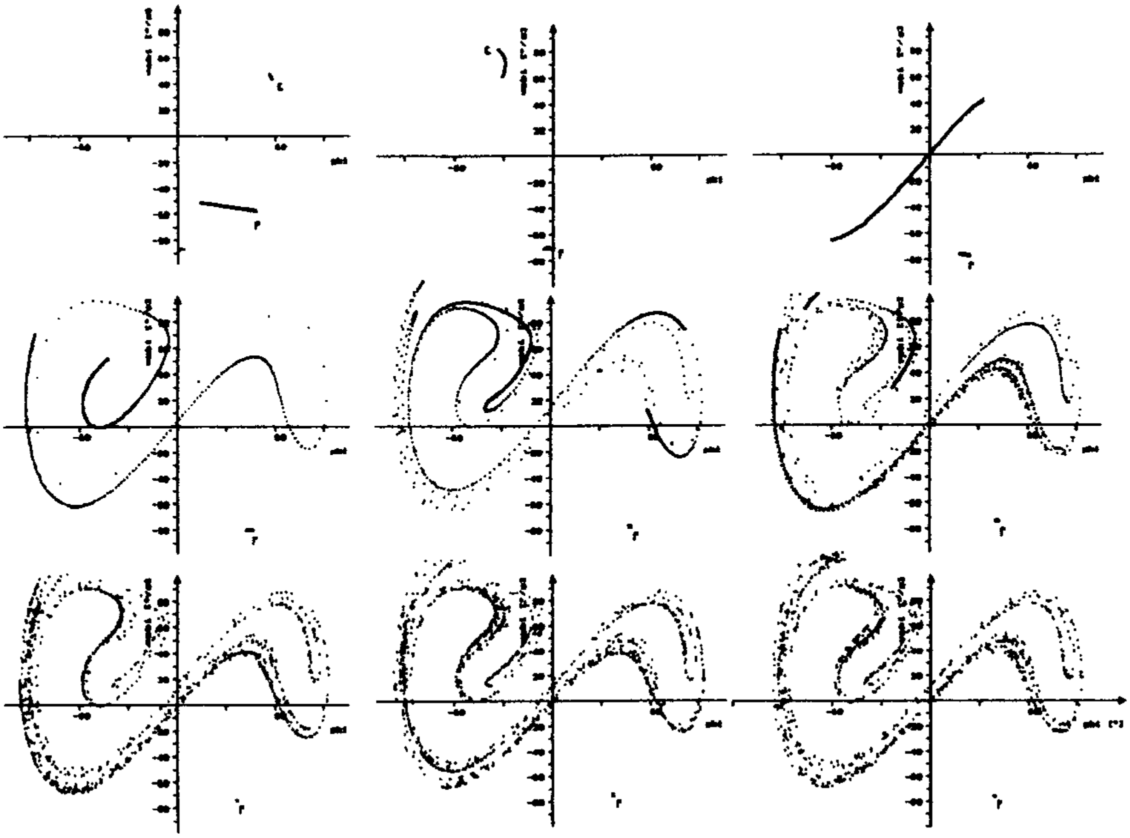

Fig. 8: Poincaré representation of the development of obits for 1000 neighboring starting points in the regular (r) and In the chaotic (c) case during the first nine periods of the drive.

The chaotic mixing machine

The idea that chaos in generated by mixing the orbits may be visualized as follows: The ensemble of the orbits in regarded at different values of the phase of the drive corresponding to displaying two dimensional slices of the above mentioned three dimensional Gugelhupf into a sequencial order (fig. 9). Letting the picture sequence run cyclically as a

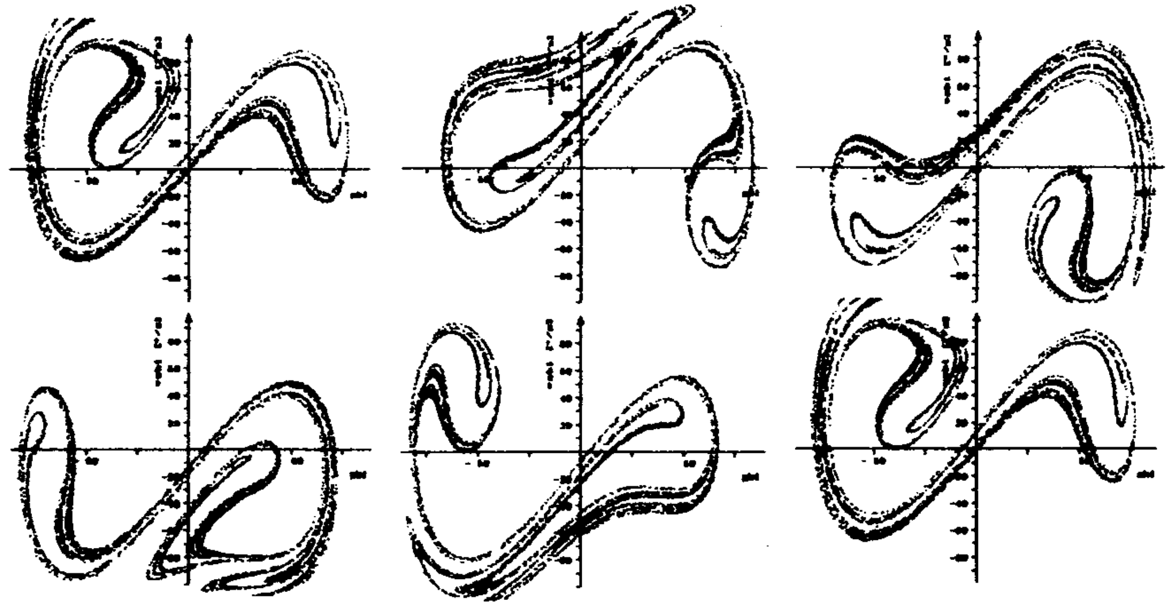

Fig. 10: : The “slices of the Gugelhupf”, i.e. the Poincaré sections at different values of the driving phase, exhibit a characteristic mixing mechanism which may be considered as origin of Chaos.

film a distinct pulsating or breathing of the attractor may be observed: As a vivid impression of the permanent interplay of the drive and the dissipation, parts of the attractor are at the same time puffed up and stretched in order to be folded together subsequently.

Fractals: Such a process of stretching and folding running down infinitely generates a structure of infinitely thin layers confined to a finite volume. The history of the attractor in contained In any part of it: Magnifications of parts of the attractor show the same layered pattern, in principle, on all scales (self similarity). As a consequence of the uniqueness of the solution of the underlying differential equation thin layers may not fuse and must stay well separated. Therefore, the attractor in a hybride object between line and area, a fractal, with a broken dimension between 1 and 2.

The horseshoe mapping: The simplest model of such a stretching and folding mechanism in Smale's horseshoe mapping, generated by sequentially stretching a square, bending it to a horseshoe and reinserting It into the original square, then, stretching it again etc. For such a horseshoe mapping the fractal dimension may easily be calculated which, in thin connection, represents a numerical measure for the “degree of chaos” of the system in question. Based on thin model there have been developed techniques to determine the fractal dimension of chaotic attractors of real systems.

Survey

In order to get a survey of the behaviour of our pendulum for the Mole range of the damping current, each time the transients have died away, the motion of the pendulum in flashed at equivalent values of the phase during many cycles of the drive. The corresponding values of the angles are plotted into a diagram (fig. 10) which shows a broad band

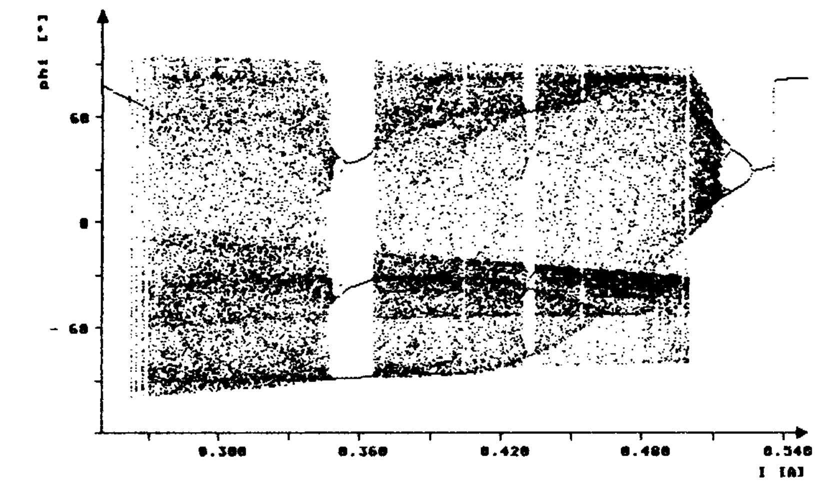

Fig. 9: Survey of the long time behaviour of the rotating pendulum on the whole range of the damping current. After transients have died away (after 100 periods of the drive) 50 values of the displacement angles are plotted when the driving phase passes 0

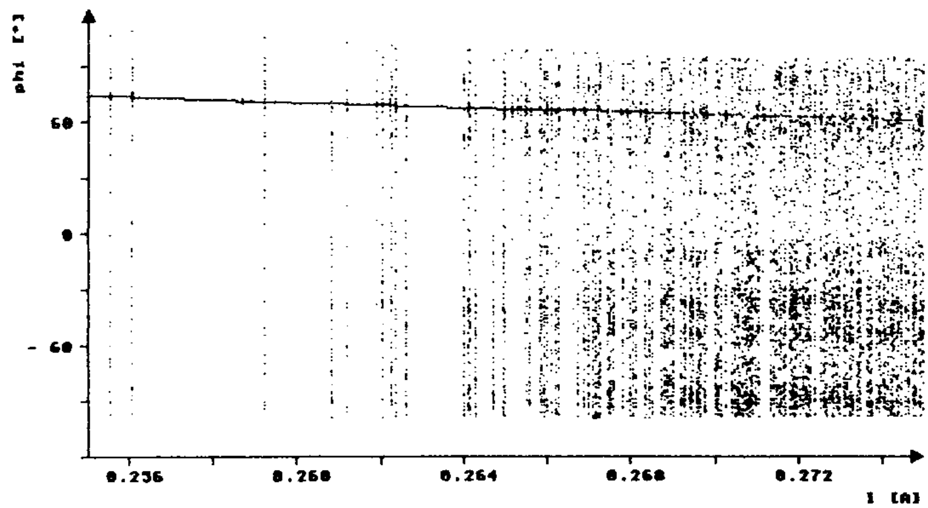

Fig. 11: Transition into chaos at small damping currents

of chaotic vibrations flanked by regular areas al low and high damping currents which occasionally in interrupted by so called windows of ordered behaviour.

Transitions to chaos

The transition from regular to chaotic behaviour, generally, takes place in a definite manner. The most Important paths into chaos can be seen in fig. 10: There are as well discontinuous breakdowns - e.g. at low damping currents and at the right edges of the windows of ordered behaviour - as continuous transitions - e.g. at great damping currents. The transition from order to chaos at small damping rates appears abruptly (fig. 10). A more *detailed investigation* of this section by varying the damping current in very small steps reveals, however, that the chaos announces itself more and more violently (fig. 11). Chaotic and ordered modes are alternating with increasing damping shifting the relation more and more in disfavor to order. (Closer inspection shows that things are still more complicated: For all the damping values shown in fig. 11 the system. finally, ends In an ordered mode, sometimes however, only after more than 2000 periods of the drive. Therefore, actually, fig. 11 reflects the alternating and diverging dying away of the transients.) Chaos casts Its shadows before when approaching the transition point.

If one considers in the same manner the right edge of the ordered window in the range 355 mA < 1 < 365 mA (see fig. 10), at every magnification, a sharp boundary between order and chaos may be detected. However, observing the stationary modes at an increasing rate of the damping (fig. 12), chaos announces Itself in thin case, too: At I = 365 mA the motion in still regular. Increasing the current to I = 366 mA the motion in occasionally interrupted by chaotic bursts which, however, rapidly die away. A further increase of the current makes these distur-

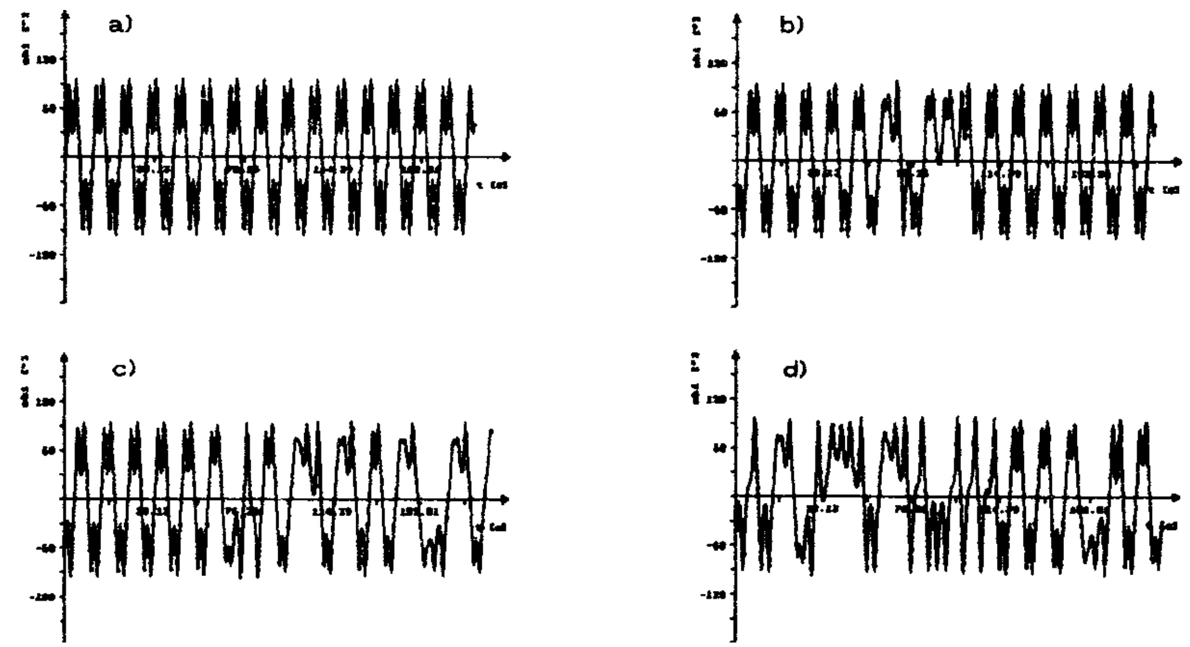

Fig. 12: Vibration modes at an intermittency transition to chaos:

a) I = 365 mA, b) I = 366 mA, c) I = 366.5 mA, d) I = 377 mA

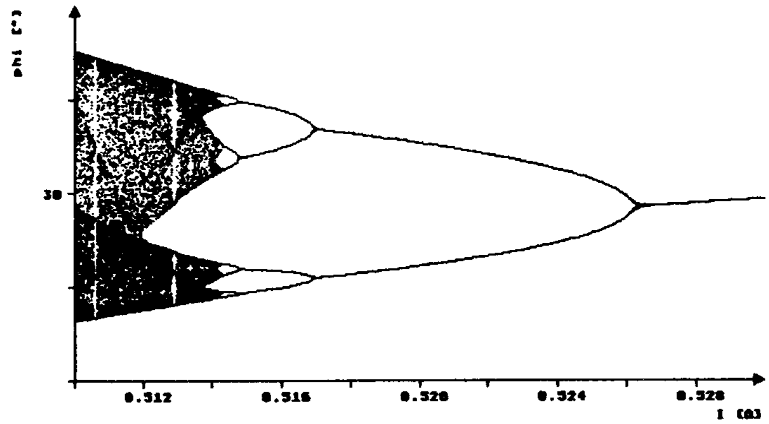

Fig. 13: : Feigenbaum transition to chaos.

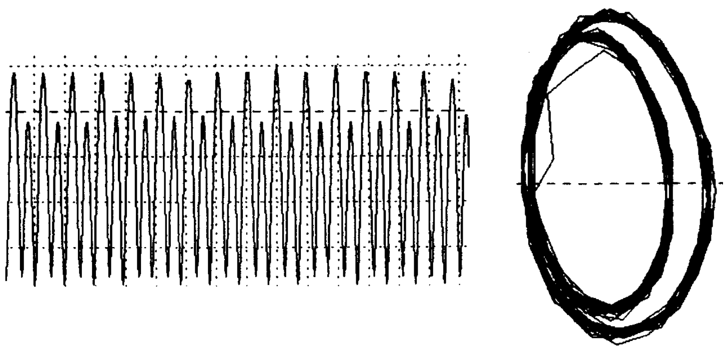

Fig. 14: First period doubling in the configuration space and phase space representation (experimentally recorded curves).

bances appear more often, and last longer and longer until, finally. at I = 367 mA the motion stays chaotic. only occasionally interrupted by short reminiscences of the Past order. In the survey diagram (fig. 10) thin reminiscence manifests itself in a shadowy continuation of the “regular” lines into the chaotic regime. Thin transition to chaos in called **intermittency.**

The best known transition to chaos in the so called **Feigenbaum-transition,** which can be observed at a high damping rate by reducing the current: The vibration with a constant amplitude In suddenly replaced by a vibration of two different amplitudes alternating regularily. After some further reduction of the damping current each of the new amplitudes splits again and so on... until finally the amplitudes are alternating in a chaotic manner. The distance between these bifurcations in decreasing according to a universal law characterized by the so called Feigenbaum constant [7]. By further reducing the damping rate, these chaotic bands, finally, overlap and fuse together, ocassionally interrupted by narrow bands of regular behaviour. The transition lines corresponding to the regular vibration modes develop again via a “Feigenbaum” (in German ”Feigenbaum” means “fig-tree”) into Chaos: Indeed. due to the above mentioned self similarity each magnification of these transition lines may be blown up to a figure which may not be discriminated from the original Feigenbaum.

The fully developed beauty of fig. 13 may be obtained by starting the motion of the pendulum at a slicely different value, e.g. at φ_0 = 10°. At the starting angle φ_0 = 0° which we used in all previous simulations the regularity of the figure In several times interrupted by a 3 rd order cycle, which obviously coexists with the modes investigated here. The Feigenbaum transition in. a universal feature which may be encountered at totally different systems like e.g. our pendulum and the logistic equation.

Summary

The investigation of chaotic systems relies essentially on qualitative, geometric techniques. A dynamical process In considered as a change of the volume characterising all possible initial conditions in phase space In course of time. Different from regular processes which also lead to a deformation of the corresponding volume in phase space, chaotic processes end in a mixing of the orbits caused by a stretching and folding mechanism.

A rotating pendulum (wellknown at German physics departments as “Pohlsches Rad” for the demonstration of forced vibrations) may be manipulated by fixing an additional mass at the oscillating wheel. Due to thin manipulation all kinds of nonlinear vibrations as e.g. regular and chaotic vibrations, coexisting vibration modes and their basins. windows of order, and the first few bifurcations of the Feigenbaum may experimentally be demonstrated at least in a qualitative way.

The derivation of the equation of motion in so simple that their numerical Integration and the discussion of the numerically obtained results will even be possible in high school physics courses. Therefore, typical techniques, concepts and results of chaos research may be discussed at a single system.

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