On optimal stopping of one-dimensional symmetric diffusions with nonlinear costs of observations

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Chapter 1

Introduction

Problems of optimal stopping arise in various fields of applied mathematics. In sequential analysis, the determination of optimal Bayes-tests leads to a problem of optimal stopping for a functional of the process of posterior distributions. In mathematical finance, pricing of American contingent claims may be reduced to solving an optimal stopping problem for a diffusion process. Furthermore in portfolio optimization, Morton and Pliska [47] introduced an appropiate stopping problem to determine an asymptotic optimal growth rate under consideration of transaction costs. In quality control, optimality of the CUSUM procedure may be derived via an optimal stopping problem, see Beibel [4], Ritov [51].

In the following, we will give a brief overview on the development of the theory of optimal stopping. This type of decision problems can in general be described by a mathematical model as follows:

Ingredients are

- a time parameter set $T \subset [0, \infty)$,
- a filtered probability space $(\Omega, (\mathcal{F}_t)_{t \in T}, P)$,
- an adapted payoff process $(Z_t)_{t \in T}$.

Sucessively one obtains informations represented by the filtration $(\mathcal{F}_t)_{t \in T}$. At each time point $t \in T$ one has the choice between immediate stopping, receiving the payoff Z_t, and continuation of the observations in the hope of obtaining a larger gain. Since the decision to stop must not depend on the future information strategies for termination of the observation procedure are the stopping times w.r.t. $(\mathcal{F}_t)_{t\in T}$. Denoting this set by $\mathcal{S}$ the problem is to maximize the expected payoff EZ_τ among all $\tau \in \mathcal{S}$ and to determine an optimal $\tau^* \in \mathcal{S}$ that attains this supremum.

Starting point for the theory of optimal stopping was the work of Wald and Wolfowitz [60], [59] in sequential statistics. In 1948 they determined Bayes-solutions for sequential tests by introducing and solving an appropriate optimal stopping problem. Snell [57] formulated the general problem in discrete time and investigated the relation to martingales and supermartingales. He showed the existence of a minimal dominating regular supermartingale $(U_t)_{t\in T}$ and proved under suitable assumptions that

$$\tau^* = \inf\{t \in T : Z_t = U_t\}$$

defines an optimal stopping time. Haggstrom [24] and Chow, Robbins [11] further developed the results of Snell by determining the form of U as

$$U_t = \operatorname*{ess\,sup}_{\tau \geq t} E(Z_\tau | F_t) \quad , t \in T,$$

which in the sequel has been called Snell-envelope.

Although the existence and a characterization of an optimal stopping time could be clarified, the explicit computation involves the determination of the above minimal dominating regular supermartingale U. For finite time points this can be done recursively backwards in time by the so called backward induction method which already appeared in [1]. In the monograph of Chow, Robbins and Siegmund [12] this is described in full detail.

In infinite time, generally an explicit construction of an optimal stopping time cannot be obtained. For the so called monotone problems Chow, Robbins [9], [10] showed that an optimal stopping time is given by

$$\tau^* = \inf\{n \in \mathbb{N}_0 : Z_n \geq E(Z_{n+1}|\mathcal{F}_n)\} \quad .$$

Starting with a paper of Dynkin [14] a theory of optimal stopping for Markov processes has been developed. With an underlying Markov process

$$X = ((X_t)_{t\in T}, (\mathcal{F}_t)_{t\in T}, (P_x)_{x\in E})$$

the payoff process has the form $Z_t = h(X_t)$ with a function $h : E \to \mathbb{R}$. Depending on the starting point the optimal expected payoff is a function $v : E \to \mathbb{R}$ defined by

$$v(x) = \sup_{\tau\in\mathcal{S}} E_x h(X_\tau)$$

for all $x \in E$. Introducing excessive and regular excessive functions, a characterization of v as minimal h dominating regular excessive function could be given, see Shiryayev [56], Theorem 8. Furthermore, the optimality of

$$\tau^* = \inf\{t \in T : h(X_t) = v(X_t)\} \tag{1.1}$$

was shown. This led to a division of the state space E into a continuation and a stopping region, denoted by $\mathcal{C}$ and $\mathcal{E}$. Thus

$$\mathcal{C} = \{x \in E : v(x) > h(x)\} \quad , \quad \mathcal{E} = \{x \in E : v(x) = h(x)\}$$

and the optimal stopping time, defined in (1.1), is the first exit time from $\mathcal{C}$. For various Markov processes the set of excessive functions could be determined, leading to explicit solutions for associated optimal stopping problems, see Dynkin, Yushkevich [15] for some examples.

The ideas for discrete time stopping problems can be used also in contionuous time. Following the martingale direction Fakeev [19], Mertens [42], [43] and Thompson [58] carried over the approach of a minimal dominating regular supermartingale from discrete to continuous time. Monotone case stopping problems in continuous time were introduced and solved in Irle [27].

For so called standard Markov processes the same could be done, initiated by Dynkin[14] and further developed by Engelbert [16], [17] , Shiryayev [55], Engelbert [18]. For solving optimal stopping problems explicitly, a free boundary value approach was introduced and led to explicit solutions for various problems, see Bather [2], p. 606 , Bather [3], Shiryayev [54], Mikhalevich [46], Shiryayev [56], Chapter 4, Mc Kean [41], von Moerbeke [44],[45] , Shiryayev [53].

In a simplified way, one may say that an explicit solution can be obtained if the underlying Markov process has a one-dimensional state space and the payoff does not depend on time. For payoffs, depending linearly on time in the form of $h(x,t) = g(x) - ct$, a reduction to an appropiate one-dimensional free boundary value problem often leads to an explicit soluion as well.

If the payoff is a more complicated function of space and time, the determination of an explicit solution usually is not possible. Here one turns to asymptotic expansions of the continuation region. Such asymptotic expansions were derived for various problems, see for example Chernoff [7],[8], Breakwell, Chernoff [6], Lai [39],[40], Bather [2]. Methods related to variational inequalities were used by Friedman [20],[21] to obtain asymptotic results.

An alternative approach to optimal stopping of one-dimensional diffusions was introduced by Salminen [52], using the Choquet representation of excessive functions.

Following the martingale direction, Jacka [32] gave a new characterization of the Snell-envelope for payoff processes, that are semimartingales, by studying local time.

Beibel, Lerche [5] found a simple method to solve some stopping problems related to mathematical finance. Their basic idea is to decompose the payoff process into a product of a martingale and a uniformly bounded process. This has been carried over to one-dimensional diffusions by Paulsen [48] using the h-transformation.

Karatzas [35] showed the significance of optimal stopping for pricing of American options. The price of the so called American perpetual put in the Black-Scholes model could be already explicitly computed by Mc Kean [41] by using a free boundary value approach. Various other methods were introduced, see Karatzas, Shreve [36], Beibel, Lerche [5]. For finite running time no explicit computation has been available until now. The structure of the continuation region could be obtained by Jacka [32]. He derived an integral equation which uniquely determines its boundary function.

Also in other fields of mathematical finance, optimal stopping has become a topic of great interest. Morton, Pliska [47] used an appropiate problem for maximizing an asymptotic growth rate in portfolio theory. This paper leads to one basic application of the theory which will be developed in this thesis.

Starting point has been the fact that various stopping problems for Markov processes have a payoff function of the form

$$h(x,t) = g(x) - ct \quad .$$

This type of payoff occurs in sequential statistics when the costs for sampling are assumed to be proportional to time, and also in the paper of Morton, Pliska [47] in portfolio optimization by using a log-utility function. But it is also reasonable to introduce a nonlinear cost function $c(t)$. In sequential statistics one can argue that, due to learning mechanisms, the cost rate should decrease which would lead to a concave instead of linear growth of costs. In the framework of Morton, Pliska [47] this could lead to a treatment of other utility functions.

At the begining were several papers by Irle [29], [30], [31] who investigated, motivated by the problems of locally best tests, optimal stopping problems for the Wiener process w.r.t. the reward function

$$(x,t) \to g(x) - c(t) \quad .$$

In Irle [29], [30] $g(x) = x^+$ was treated for different typ of cost functions. As a first result it was clarified, how the continuation region can be enscribed between two boundary curves. These methods were then extended to obtain analogous results for various typs of reward functions g, see Irle [31]. But in those papers an asymptotic expansion of the boundary function of the continuation region was not obtained. This was firstly done in Kubillus, Irle, Paulsen [37] who derived such an expansion for payoff functions of the form

$$(x,t) \to x^+ - c(t)$$

with increasing concave or convex c satisfying mild additional assumptions.

It is the topic of this thesis to derive asymptotic expansions for rewards of the general form

$$(x,t) \to g(x) - c(t) \quad ,$$

not only for the Wiener process but for a wide class of diffusions generated by a second order elliptic differential operator. It will provide a new approach to handle such type of optimal stopping problems. Several applications will be presented which justify the hope that the presented methods will give a powerful tool in the asymptotic analysis of optimal stopping problems.

The main idea is to determine an inner and outer approximation of the continuation region, being asymptotically equivalent. In the case of concave c, an inner approximation can be easily derived by applying the results for linear cost functions. It is more difficult to find a suitable outer approximation. This will be done by defining a majorant $\Phi : E\times(0,\infty) \to I\!R$ of the payoff $g(x)-c(t)$, touching it at two curves $m \pm \beta_+(t)$. Additionally, Φ is superharmonic for large t, i.e.

$$(\partial_t + A)\Phi(x,t) \le 0 \quad , \quad x \in E, t \ge t_0$$

for some $t_0 > 0$, with A denoting the differential generator of the diffusion. These properties then can be used to show that the continuation region after t_0 is contained in the region enscribed between the boundary curves $(m \pm$

$\beta_+(t))_{t\geq t_0}$. Note, that symmetry of the diffusion and the reward g w.r.t. a midpoint $m \in E$ has to be supposed.

For convex c, an application of the results for linear costs leads to an outer approximation of the continuation region. To determine an inner approximation a majorant Φ will be defined, touching the payoff at two curves $m \pm \zeta_-(t)$. Here, Φ is subharmonic for large t, i.e.

$$(\partial_t + A)\Phi(x,t) \geq 0 \quad , \quad x \in E, t \geq t_0$$

for some $t_0 > 0$. With these properties we can prove that the region enscribed between the curves $(m \pm \zeta_-(t))_{t\geq t_0}$ is contained in the continuation region, providing the desired inner approximation.

We emphasize that we can formulate rather general conditions, which lead to an asymptotic expansion for the boundary of the continuation region. Two examples from sequential statistics and one from portfolio optimization will accompany us through this thesis and indicate that it is worth to study problems of optimal stopping for nonlinear costs of observations.

The thesis is organized as follows. In Chapter 2 we recall some basic facts on one-dimensional diffusions and introduce three examples of optimal stopping problems that will accompany us through this thesis. The easier case of linear costs of observations will be treated in Chapter 3. The obtained results play an important role for the analysis of the nonlinear problem. For concave and convex costs of observations the principal shape of the continuation region can be derived and an asymptotic expansion of its boundary function will be given in Chapter 4 and 5. Finally in Chapter 6 we weaken our assumptions such that a wider class of payoff functions can be treated.

Each chapter ends with an application of the obtained results to examples coming from sequential statistics and financial mathematics.

Chapter 2

Preliminaries

2.1 Diffusions

In this section, we want to clarify what we will understand under a one-dimensional diffusion generated by a differential operator A. We fix an open interval E of $\mathbb{R}$ which shall become the state space and consider at first a non terminating, strong Markov process with continuous paths

$$X = ((X_t)_{t\geq 0}, (\mathcal{F}_t)_{t\geq 0}, (P_x)_{x\in E})$$

defined on a measurable space $(\Omega, \mathcal{F})$ governed by a right continuous filtration $(\mathcal{F}_t)_{t\geq 0}$, see Shiryayev [56] p.18, Karatzas, Shreve [34], Def. 6.3, p.81 for a definition. As discussed in Rogers, Williams [50], p. 110, this class of processes is too wide and embraces examples with unruly behaviour. So we restrict our attention to strong Markov processes on E with continuous paths that are generated by stochastic differential equations. As Ikeda, Watanabe [25], p. 188, pointed out this class of processes is known to be sufficiently wide both in theory and applications.

To be more precise, we consider an elliptic differential operator

$$A = \frac{1}{2}a^2(x)\partial_x^2 + b(x)\partial_x \tag{2.1}$$

with continuous dispersion and drift functions $a, b : E \to \mathbb{R}$. We assume that the nondegeneracy condition

$$a^2(x) > 0 \qquad \text{for all } x \in E \tag{2.2}$$

holds, and can then define the scale function

$$s(x) = \int_m^x \exp(-2\int_m^y \frac{b(z)}{a^2(z)}dz)dy \tag{2.3}$$

for all $x \in E$, where m is an arbitrary chosen element of E. The function s is twice continuously differentiable, strictly increasing and satisfies $As = 0$ on E. We introduce the function $u : E \to I\!R$ by

$$u(x) = \int_m^x s'(y) \int_m^y \frac{2}{s'(z)a^2(z)}dzdy \tag{2.4}$$

for all $x \in E$. It is obvious that the function u is twice continuously differentiable and satisfies

$$Au = 1$$

on E, subject to the boundary conditions $u(m) = u'(m) = 0$. We assume that the diffusion is non-exploding. Due to Feller's test of explosion this means that $u(x)$ tends to infinity when x converges to a boundary point of E; see Theorem 5.29, Karatzas, Shreve [34].

We combine these two objects, the differential operator A and the Markov process X, in the following definition.

2.1.1 Definition: *Let the elliptic differential operator A fulfill the preceding assumptions. Then a triple*

$$X = ((X_t)_{t\geq 0}, (\mathcal{F}_t)_{t\geq 0}, (P_x)_{x\in E})$$

is called an A-diffusion with state space E if

(i) X is a strong Markov process with continuous paths on E and

(ii) for each $f \in C_K^2(E)$ the process

$$(f(X_t) - f(X_0) - \int_0^t Af(X_s)ds)_{t\geq 0}$$

defines a P_x-martingale w.r.t. $(\mathcal{F}_t)_{t\geq 0}$ for all $x \in E$.

Here and in the following $C_K^2(E)$ denotes the space of twice continuously differentiable functions with compact support contained in E. Condition (ii) states that $C_K^2(E)$ is contained in the domain of the infinitesimal generator of X, which coincides on $C_K^2(E)$ with the elliptic differential operator A. For the following we want to denote by C^2 the space of twice continuously differentiable functions.

For the purpose of optimal stopping, the martingale introduced in (ii) can be used to derive the following formula, which will be used in this thesis several times.

2.1.2 Proposition *Let X be an A-diffusion with state space E. Let $U, V \subset E$ be open sets such that $\bar{V}$ is bounded and contained in U. Let furthermore $g : U \to I\!R$ be a C^2-function and $\tau = \inf\{t \geq 0 : X_t \notin V\}$ the first exit time from V.*
Then for each stopping time σ

$$E_x g(X_{\tau\wedge\sigma}) = g(x) + E_x \int_0^{\tau\wedge\sigma} Ag(X_s)ds \tag{2.5}$$

for all $x \in V$.

Proof: The claim follows immediately with optional stopping, since there exists an $f \in C_K^2(E)$ that conicides on V with g. Hence

$$(g(X_{\tau\wedge t}) - g(x) - \int_0^{\tau\wedge t} Ag(X_s)ds)_{t\geq 0}$$

is a bounded martingale. □

The main examples that we are going to consider bring together the concept of martingale problems and stochastic differential equations. For each $x \in E$ we consider the stochastic differential equation

$$dY_t = a(Y_t)dW_t + b(Y_t)dt \quad , \quad Y_0 = x, \tag{2.6}$$

with W denoting a standard Wiener process. Since the coefficients fulfill the nondegeneracy and local integrability condition, the above equation (2.6) admits a weak solution up to an explosion time and this solution is unique in the sense of probability law, see Theorem 5.15 of Karatzas, Shreve [34]. Due to Feller's test, an explosion cannot occur and a probability measure Q_x on $(C([0,\infty), E), \mathcal{B}_\infty)$ can be uniquely defined by the law of a solution of equation (2.6). Here $C([0,\infty), E)$ denotes the space of continuous functions from $[0,\infty)$ to E, endowed with the Borel σ-field $\mathcal{B}_\infty$.

Let us denote the coordinate process by $X_t(\omega) = \omega(t)$ for $\omega \in C([0,\infty), E)$, $t \geq 0$, and by $\mathcal{B}_t$ the σ-field generated by $\{X_s : s \leq t\}$. Then, from a weak solution of (2.6), we obtain with Ito's formula that

$$f(X_t) - f(X_0) - \int_0^t Af(X_s)ds$$

is a Q_x martingale w.r.t. $(\mathcal{B}_{t+})_{t\geq 0}$ for each $x \in E$ and $f \in C_K^2(E)$. This shows that Q_x is the unique solution to the corresponding martingale problem, see Rogers, Williams [50] V.19.

The assumptions made on A imply that the corresponding martingale problem is well-posed. This is equivalent to the well-posedness of the corresponding stochastic differential equation (2.6). This fact furthermore implies that the familiy of distributions $(Q_x)_{x\in E}$ together with the coordinate process $(X_t)_{t\geq 0}$ and filtration $(\mathcal{B}_{t+})_{t\geq 0}$ defines a strong Markov process, giving an A-diffusion in canonical representation.

Throughout this thesis, only symmetric diffusions will be treated.

2.1.3 Definition: *An A-diffusion X is called symmetric w.r.t. a midpoint $m \in E$, if the following holds:*

(i) $E = (m-l, m+l)$ *with* $l \in (0,\infty]$,

(ii) $a(m+y) = a(m-y), b(m+y) = -b(m-y)$ *for all* $y \in (0,l)$.

It is called mean reverting if $b(m+y) \leq 0$ *for all* $y \in (0,l)$.

Note that the case $l = \infty$ is included in the definition and corresponds to $E = \mathbb{R}$. In this case, different to bounded E, multiple choice of a midpoint may be possible.

Mean reverting diffusions have the tendency to pull back to the midpoint. For us, a further property is of interest and will be used in the following chapters.

2.1.4 Proposition *Let X be a symmetric, mean reverting A-diffusion w.r.t. $m \in E$. Then the unique solution u of $Au = 1$ subject to $u(m) = u'(m) = 0$ is a strictly convex function, even w.r.t m.*

Proof: Due to (2.3),(2.4) and the symmetry conditions, the function u is even w.r.t. m, i.e. $u(m+y) = u(m-y)$ for all $y \in (0,l)$. Furthermore, due to $Au = 1$, u satisfies

$$u''(x) = 2\frac{1 - b(x)u'(x)}{a^2(x)}$$

for all $x \in E$. Since X is mean reverting the right-hand side is strictly positive and thus the assertion is valid. □

The symmetry conditions for the coefficients of A leads to a symmetry property for the laws of the corresponding diffusion.

2.1.5 Proposition *Let X be a symmetric A-diffusion w.r.t. $m \in E$. Then the law of $2m - X$ w.r.t P_x coincides with the law of X w.r.t. P_{2m-x} for all $x \in E$.*

Proof: The result follows from uniqueness in the corresponding martingale problem. As well one may argue with uniqueness of law of a weak solution of (2.6); see Rogers, Williams V.19 [50]. To be precise, we examine that

$$f(2m - X_t) - f(2m - X_0) - \int_0^t Af(2m - X_s)ds$$

is a P_x-martingale for all $f \in C_K^2(E)$. This follows from the symmetry condition (ii), if we define the one-to-one map $\phi : C_K^2(E) \to C_K^2(E)$, $\phi(f)(x) = f(2m-x)$ for all $x \in E$. Then for $g = \phi(f)$ we have $Ag(x) = Af(2m - x)$ for all $x \in E$, and the above martingale property follows from that of an A-diffusion. This implies that the law of $2m - X$ w.r.t. P_x is a solution to the martingale problem with initial point $2m - x$. Since this solution is unique, it must coincide with the law of X w.r.t. P_{2m-x} and the assertion is shown. □

2.2 Main examples

We introduce three examples coming from different areas of probability theory and statistics that will accompany us throughout this thesis.

2.2.1 Brownian motion

The basic diffusion is Brownian motion. It is an A-diffusion with $E = I\!R$ and $A = \frac{1}{2}\partial_x^2$. Furthermore, it is symmetric and mean reverting w.r.t. each $m \in I\!R$. For simplicity we choose the origin as midpoint. Then the scale function defined in (2.3) and the symmetric solution of $Au = 1$ vanishing at zero are given by

$$s(x) = x \quad , \quad u(x) = x^2$$

for all $x \in I\!R$.

For our purpose of optimal stopping with nonlinear costs, it is important to note that

$$E_x \sup_{t \geq 0}(|X_t|^\alpha - t^\beta) < \infty \tag{2.7}$$

for all $x \in I\!R$, if $\alpha > 0$ and $\beta > \alpha/2$, see Irle [31], Chow and Teicher [13], Ch. 10.4.

2.2.2 Process of posterior probabilities

This is an example of an A-diffusion with state space $E = (0, 1)$ and differential generator

$$A = \frac{1}{2}x^2(1-x)^2\partial_x^2 \quad . \tag{2.8}$$

It arises in sequential statistics when testing the drift of a Wiener process, see Shiryayev [56], Ch. 4.2, as we will briefly discuss in the following.

Let $(W_t)_{t \geq 0}$ be a standard Wiener process, that is a Browinan motion starting from the origin, defined on a probability space $(\Omega, \mathcal{F}, P_\pi)$ and Θ a random variable, independent of the Wiener process and satisfying

$$P_\pi(\Theta = 1) = \pi = 1 - P_\pi(\Theta = 0)$$

for some $\pi \in (0, 1)$. One observes the stochastic process

$$\zeta(t) = r\Theta t + \sigma W_t \qquad , \qquad \sigma^2 > 0, r \neq 0. \tag{2.9}$$

Thus, given $\Theta = 1$, the observed process behaves like a Wiener process with drift r, whereas ζ is a Wiener process without drift given $\Theta = 0$. The statistical problem consists of testing the hypothesis of no drift against the alternative of drift r, based upon the observations obtained by ζ. As usual in Bayesian statistics, a reasonable procedure relies on the process of posterior probabilities $(\Pi_t)_{t\geq 0}$, defined by

$$\Pi_t = P_\pi(\Theta = 1|\mathcal{F}_t^\zeta)$$

for all $t \geq 0$ with $\mathcal{F}_t^\zeta = \sigma(\{\zeta_s, s \leq t\})$. Π_t denotes the posterior probability of the alternative, given the information up to time t. An application of the theorem of Girsanov provides the existence of a standard Wiener process $\bar{W}$ on $(\Omega, \mathcal{F}, P_\pi)$ such that $(\Pi_t)_{t\geq 0}$ satisfies

$$d\Pi_t = \frac{r}{\sigma}\Pi_t(1-\Pi_t)d\bar{W}_t \quad , \quad \Pi_0 = \pi \quad . \tag{2.10}$$

For simplicity, we assume $\frac{r}{\sigma} = 1$. Then the family of laws of $(\Pi_t)_{t\geq 0}$ w.r.t. P_π, when π runs through $(0,1)$, constitutes an A-diffusion with generator (2.8) in canonical representation.

It is symmetric w.r.t $m = \frac{1}{2}$ and mean reverting. Since there is no drift, we have a natural scale. The unique symmetric solution of $Au = 1$, vanishing at m, fulfills

$$u(x) = 2(2x-1)\log\frac{x}{1-x} \quad , \quad u'(x) = 4\log\frac{x}{1-x} + \frac{2(2x-1)}{x(1-x)} \quad . \tag{2.11}$$

We note that the assumptions on A are satisfied, since $u(x)$ tends to infinity when x converges to one of the boundary points.

2.2.3 Portfolio optimization

We will introduce an A-diffusion with state space $E = (0,1)$ and generator

$$A = \frac{1}{2}x^2(1-x)^2\partial_x^2 + x(1-x)(\frac{1}{2}-x)\partial_x \quad , \tag{2.12}$$

and furthermore give its relation to mathematical finance, in particular portfolio optimization.

An obvious question for a capital investor is how to divide his money into a risk-free bank account and a risky asset like a stock. In the Black-Scholes model, the price process $(S_t)_{t\geq 0}$ of the risky asset follows a geometric Brownian motion,

$$dS_t = S_t(\mu dt + \sigma dW_t) \quad , t \geq 0. \tag{2.13}$$

The positive constants σ, μ denote the volatility and rate of expected return respectively. One unit of money increases in the risk-free bank account as $(e^{rt})_{t\geq 0}$ with r denoting the constant interest rate. Obviously, one may suppose $\mu > r$. This means that, on average, the risky asset has larger growth than the risk-free security.

Starting from a fraction π_0 of the initial capital x, we invest $x\pi_0$ in the stock and put $(1-\pi_0)x$ into the bank account. Then we have the ability to trade continuously in a self-financing manner. This means, that no extra money can be invested from outside and no money can be withdrawn. This kind of trading strategy is uniquely determined by the evolution of the fraction of wealth held in the risky asset, denoted by $(\pi_t)_{t\geq 0}$. The so-called trading or portfolio startegies are linked together with their corresponding wealth process $(V_t)_{t\geq 0}$ via the stochastic differential equation

$$dV_t = V_t((1-\pi_t)rdt + \pi_t(\mu dt + \sigma dW_t)) \quad , \quad V_0 = x. \tag{2.14}$$

We set $\hat{b} = \frac{\mu - r}{\sigma^2}$ and assume $\hat{b} \in (0,1)$. The high volatility of the stock compensates the higher return μ compared to r. It is well known that, for a given time horizon T, the portfolio strategy $\pi_t^* \equiv \hat{b}, 0 \leq t \leq T$, maximizes the expected return $(E \log V_T)/T$ among all portfolio strategies π; see Karatzas, Shreve [36]. The maximal obtained value, independent of the time horizon, is given by

$$R^* = (1-\hat{b})r + \hat{b}\mu - \frac{1}{2}\hat{b}^2\sigma^2 = r + \frac{1}{2}\hat{b}(\mu - r) \quad . \tag{2.15}$$

This strategy requires that an investor has to change the number of shares of the risky asset continuously to stay at the optimal point $\hat{b}$ of balance. This causes substantial transaction and management costs which may not be desirable. To avoid these costs, the following procedure seems reasonable.

Starting with an initial fraction π_0, we consider that portfolio strategy that forbids any transaction. This means that the number $c = \pi_0 V_0 / S_0$ of shares of the stock will be held constant over time. The corresponding portfolio strategy π satisfies $\pi_t = cS_t/V_t$. Due to (2.14), it fulfills

$$d\pi_t = \pi_t(1-\pi_t)(\sigma^2(\hat{b} - \pi_t)dt + \sigma dW_t) \quad . \tag{2.16}$$

Thus (π_t) evolves like an A-diffusion if $\hat{b} = \frac{1}{2}$ and $\sigma = 1$. The assumption $\sigma = 1$ is only made for simplicity and not necessary. The condition $\hat{b} = \frac{1}{2}$ is restrictive but has to be supposed to obtain symmetry for the A-diffusion.

We note that in this symmetric case the diffusion also is mean reverting with scale function $s(x) = \log(x/(1-x))$ for all $x \in (0,1)$. Furthermore, the unique symmetric solution of $Au = 1$, vanishing at m, satisfies

$$u(x) = \log(\frac{x}{1-x})^2 \quad , \quad u'(x) = 2\log(\frac{x}{1-x})\frac{1}{x(1-x)} \tag{2.17}$$

for all $x \in (0,1)$.

The following transformation will be useful.

2.2.4 Proposition *Let X be a weak solution of*

$$dX_t = X_t(1-X_t)((\frac{1}{2} - X_t)dt + dW_t) \quad , \quad X_0 = x.$$

Then the law of $(\log(\frac{X_t}{1-X_t}))_{t\geq 0}$ coincides with that of a Wiener process starting from $\log\frac{x}{1-x}$.

Proof: Using Ito's formula, we obtain

$$d\frac{X_t}{1-X_t} = \frac{X_t}{1-X_t}(\frac{1}{2}dt + dW_t) \quad , \quad \frac{X_0}{1-X_0} = \frac{x}{1-x}.$$

Applying the logarithm yields the assertion. □

From this, we obtain the following corollary which will be used frequently.

2.2.5 Corollary: *Let A be the differential generator defined in (2.12), and let X be an A-diffusion on $E = (0,1)$. Then for all $\alpha, \beta > 0$ such that $\beta > \frac{\alpha}{2}$*

$$E_x(\sup_{t\geq 0} |\log\frac{X_t}{1-X_t}|^\alpha - t^\beta) < \infty \tag{2.18}$$

for all $x \in E$.

Proof: Since $\log\frac{X_t}{1-X_t}$ is a Wiener process, the assertion follows immediately from (2.7). □

2.3 Optimal stopping

We consider an A-diffusion

$$X = ((X_t)_{t\geq 0}, (\mathcal{F}_t)_{t\geq 0}, (P_x)_{x\in E}) \quad ,$$

introduced in Chapter 2.1.1 and want to treat related problems of optimal stopping.

Associated with the diffusion X is a payoff function

$$h : E \times [0, \infty) \to I\!R; (x, t) \to h(x, t) \tag{2.19}$$

The value $h(x, t)$ will be interpreted as the payoff for stopping the diffusion at time point t in state x. Starting from x_0 at time point t_0, we observe the diffusion and have at each $t \geq 0$ the choice between immediate stopping, receiving the payoff $h(X_t, t_0 + t)$, and continuation of the observations in the hope of obtaining a larger gain.

Of course, our decision to stop must not depend on the future behaviour of our observed process. Thus our strategies consist of the set of all $(\mathcal{F}_t)_{t\geq 0}$ stopping times τ such that the expected payoff

$$E_{x_0} h(X_\tau, t_0 + \tau) \tag{2.20}$$

exists. Denoting this set of stopping times by $\mathcal{S}$, the problem is to determine an optimal strategy τ^* and the optimal value

$$v(x_0, t_0) = \sup_{\tau\in\mathcal{S}} E_{x_0} h(X_\tau, t_0 + \tau) \tag{2.21}$$

In many situations, we can describe the optimal strategy in the following way. Let us define the sets

$$\mathcal{C} = \{(x, t) : v(x, t) > h(x, t)\} \tag{2.22}$$

and

$$\mathcal{E} = \{(x, t) : v(x, t) = h(x, t)\} \quad . \tag{2.23}$$

Then $\mathcal{C}$ is called the region of continuation, whereas its complement $\mathcal{E}$ is called early exercise region or stopping region. Intuitively one would expect that, starting from x_0 at time point t_0, the strategy

$$\tau^*_{t_0} \;=\; \inf\{t \geq 0 : v(X_t, t_0 + t) = h(X_t, t_0 + t)\}$$

$$= \inf\{t \geq 0 : (X_t, t_0 + t) \notin \mathcal{C}\} \tag{2.24}$$

is optimal. This is indeed the case in many situations, see Shiryayev [56], Lai [39].

Often, in particular in sequential statistics, the payoff function has the form

$$h(x,t) = g(x) - c(t) \qquad \text{for all } x \in E, t \geq 0.$$

The function g measures the reward one obtains from the states of the diffusion, whereas the non-negative function c indicates the costs which arise from the observation of the process. This typ of payoff functions will be treated in this thesis.

In the following, we will discuss various optimal stopping problems.

2.3.1 Locally best tests

We briefly repeat some results from Irle [26],[28] and give its relation to our context for the problem of testing the drift of a Wiener process.

Let $(\Omega, \mathcal{F})$ be a measurable space with a right continuous filtration $(\mathcal{F}_t)_{t\geq 0}$ and let $(P_\theta)_{\theta \geq 0}$ be a family of probability measures on $(\Omega, \mathcal{F})$ such that an observed process $(W_t)_{t\geq 0}$ is a Wiener process with drift θ w.r.t. P_θ for all $\theta \geq 0$. We consider the problem of sequentially testing the hypothesis $H = \{0\}$ against the alternative $K = \{\theta : \theta > 0\}$. For a sequential test (τ, ψ), consisting of a stopping time τ and a terminal $\mathcal{F}_\tau$-measurable decision function ψ, the power function $\beta(\tau, \psi)$ is defined by

$$\beta(\tau,\psi)(\theta) = E_\theta \psi 1_{\{\tau<\infty\}} = E_0 \psi \exp(\theta W_\tau - \frac{1}{2}\theta^2 \tau) 1_{\{\tau<\infty\}}$$

for all $\theta \geq 0$.

According to a given error probability α and a given bound $T > 0$, the concept of locally best tests consists of maximizing the slope of the power function at zero among all sequential tests with the same error probability α and an expected sample size not exceeding T.

As a measure for the slope, motivated by formal differentiation under the integral, we define

$$\lambda(\tau, \psi) = E_0 \psi W_\tau \quad .$$

Clearly, we can restrict our attention to sequential tests with finite expected sample size.

We consider for $m \in I\!R$ and $c > 0$ with $m - \frac{1}{4c} < 0 < m + \frac{1}{4c}$ the optimal stopping problem for

$$((W_t - m)^+ - ct)_{t \geq 0} \tag{2.25}$$

w.r.t. P_0. The optimal stopping time is given by

$$\sigma_{m,c} = \inf\{t \geq 0 : |W_t - m| > \frac{1}{4c}\} \quad ,$$

see Irle [26]. We define the decision function

$$\psi_{m,c} = 1_{\{W_{\sigma_{m,c}} \geq m + \frac{1}{4c}\}}$$

and obtain

$$\lambda(\tau, \psi) \leq \lambda(\sigma_{m,c}, \psi_{m,c})$$

for all sequential tests (τ, ψ) which satisfy

$$\beta(\tau, \psi)(0) = \beta(\sigma_{m,c}, \psi_{m,c})(0) \quad , \quad E_0\tau \leq E_0\sigma_{m,c} \quad .$$

Thus, if we choose m, c such that

$$\begin{aligned} E_0\sigma_{m,c} &= (\frac{1}{4c})^2 - m^2 = T \quad , \\ E_0\psi_{m,c} &= P_0(W_{\sigma_{m,c}} \geq m + \frac{1}{4c}) = \frac{\frac{1}{4c} - m}{\frac{1}{2c}} = \alpha \quad , \end{aligned}$$

then $(\sigma_{m,c}, \psi_{m,c})$ defines a locally best test w.r.t. error probability α and expected sample size T.

The forgoing approach relies on the assumption that costs for observations grow linearly in time. But due to learning mechanisms it is also reasonable to assume that the cost rate decreases, and this leads to a concave cost function. Hence, inserting in (2.25) a general nonlinear cost function, leads to a stopping problem of the type we are going to investigate in this thesis, and a solution of the correspong stopping problem leads to an alternative locally best sequential test. Note that the stopping problem with payoff $(x, t) \to x^+ - c(t)$ is equivalent to that for the symmetric payoff

$$(x, t) \to |x| - c(t) \quad ,$$

see [31].

2.3.2 Bayes tests

In general, the theory of optimal stopping arises in sequential statistics when determining an optimal Bayes-test. For the problem of testing the drift of a Wiener process, we want to establish the associated optimal stopping problem.

We recall the notations of 2.2.2 and note that the Bayes-risk of a sequential test (τ, ψ), based upon the observation process $\zeta_t = r\Theta t + \sigma W_t$, is defined by

$$\begin{aligned} B((\tau,\psi)) &= E_\pi(\Pi_\tau 1_{\{\psi=0\}} + (1-\Pi_\tau)1_{\{\psi=1\}} + c\tau) \\ &\geq E_\pi(\min\{\Pi_\tau, 1-\Pi_\tau\} + c\tau) \quad . \end{aligned}$$

For given stopping time τ, we choose the decision rule

$$\psi^* = 1_{\{\Pi_\tau \geq 1-\Pi_\tau\}} \quad ,$$

and obtain a Bayes test by minimizing $E_\pi(\min\{\Pi_\tau, 1-\Pi_\tau\} + c\tau)$ among all $(\mathcal{F}_t^\zeta)$ stopping times. This minimization problem is equivalent to the optimal stopping problem w.r.t. the A-diffusion introduced in 2.2.2 with payoff function

$$h(x,t) = |x - \frac{1}{2}| - ct \quad .$$

As in the preceding example, a replacement of linear costs by concave or convex costs leads to a corresponding optimal Bayes test when solving the modified optimal stopping problem.

2.3.3 Portfolio optimization

We continue the discussion of 2.2.3. The optimal portfolio strategy for maximizing the expected return consists of holding the constant fraction $\hat{b} = \frac{\mu-r}{\sigma^2}$ of wealth in the risky asset. This strategy yields an optimal growth rate $R^* = r + \frac{1}{2}\hat{b}(\mu - r)$, but has the disadvantage of causing non neglegible transaction costs as continuous trading is necessary. To avoid transaction costs, the following procedure seems reasonable, as mentioned by Morton, Pliska [47]. Starting from the initial fraction $\hat{b}$, the number $c = \hat{b}V_0/S_0$ of shares of the risky asset will be held constant over time as long as the fraction of wealth in stock is not too far away from $\hat{b}$. When the departure from $\hat{b}$ exceeds a certain level, we rebalance our portfolio to $\hat{b}$.

The question on hand is, how this random non anticipating time point for rebalancing should be chosen. One reasonable strategy is the following. We compare at each $t \geq 0$ the return $\log V_t$ of the trading strategy without transactions to R^*t, the optimal expected return, receiving from portfolio strategies that allow transactions. Thus we consider the payoff process

$$\log V_t - R^*t \quad , t \geq 0 \quad .$$

We solve the corresponding optimal stopping problem and use its solution τ^* as rebalancing time point for our portfolio.

Since the wealth process is associated with the evolution (π_t) of the fraction in stock by $(1-\pi_t)V_t = (1-\hat{b})V_0 e^{rt}$, the optimal stopping problem is equivalent to that one for an A-diffusion with generator (2.12) according to the payoff function

$$h(x,t) = \log \frac{1}{1-x} - (R^* - r)t \quad .$$

Instead of taking linear growth from the optimal strategy a generalization would be obtained by assuming a nonlinear one and solving the corresponding optimal stopping problem.

Chapter 3

Linear costs of observations

3.1 General theory

We consider optimal stopping problems for one-dimensional diffusions with payoff functions h of the form

$$h(x,t) = g(x) - ct$$

with a positive cost constant $c > 0$. As was mentioned in the introduction, these problems were often investigated by a free boundary value approach. For specific payoffs the continuation region, giving by two straight line boundaries, could be obtained, see Shiryayev [56], Ch. 4 , Morton, Pliska [47]. We want to adjust the approach of Beibel, Lerche [5] to symmetric A-diffusions. This results in sufficient conditions for the reward function g providing continuation regions of the above form, see Theorem 3.1.1. These results are necessary for the analysis of the stopping problem with nonlinear costs leading to inner, and outer approximations respectively, as we will see in the following chapters.

We recall the notations and assumptions from the preceding chapter. Thus X denotes an A-diffusion with state space E and differential generator

$$A = \frac{1}{2}a^2(x)\partial_x^2 + b(x)\partial_x \quad . \tag{3.1}$$

We assume the symmetry conditions

$$a(m+y) = a(m-y) \quad , \quad b(m+y) = -b(m-y) \quad \text{for all } y \in (0,l).$$

The real number m denotes the midpoint of $E = (m-l, m+l)$ and $2l$ its length. $E = I\!R$ is covered to $l = \infty$. An essential role in the analysis of the linear stopping problem is played by the symmetric solution u of $Au = 1$. Since X is non-exploding u tends to infinity at the boundary of E. For symmetric reward functions g and positive constants c we will study the problem of optimal stopping for the payoff function

$$(x,t) \to g(x) - ct \quad .$$

We define the optimal value by

$$v(x) = \sup_{\tau \in \mathcal{S}} E_x \left(g(X_\tau) - c\tau \right) \tag{3.2}$$

for all $x \in E$, with $\mathcal{S}$ denoting the set off all stopping times. The problem is to determine an optimal one that attains this supremum. We introduce functions $G, U : [0,l) \to [0,\infty)$ by

$$G(y) = g(m+y) \quad , \quad U(y) = u(m+y) \tag{3.3}$$

for all $y \in (0,l)$. We state the following conditions, which will lead to straight line boundaries for the optimal continuation region.

(L1) For each $x \in E$ there exists some $\varepsilon > 0$ such that

$$E_x \sup_{t \geq 0} (g(X_t) - (c-\varepsilon)t) < \infty \quad .$$

(L2) $y \to G(y) - cU(y)$ has a unique maximum at $\psi(c) \in (0,l)$.

(L3) g is C^2 on $(m + \psi(c) - \varepsilon, m + l)$ for some $\varepsilon > 0$ and fulfills $Ag(x) \leq c$ for all $|x - m| > \psi(c)$.

The main task is to prove that the continuation region defines an open interval.

3.1.1 Theorem: *Let g be a symmetric continuous reward function bounded from below, and let $c > 0$. Assume conditions (L1)-(L3).*
Then the continuation region $\mathcal{C}$ coincides with the open interval $\mathcal{C} = (m - \psi(c), m + \psi(c))$ and

$$\tau^* = \inf\{t \geq 0 : |X_t - m| \geq \psi(c)\}$$

defines an optimal stopping time. Furthermore the optimal value function v fulfills

$$v(x) = \begin{cases} cu(x) + (g - cu)(m + \psi(c)) & \text{if } |x - m| < \psi(c) \\ g(x) & \text{if } |x - m| \geq \psi(c) \end{cases} \tag{3.4}$$

for all $x \in E$.

For the proof we use the following Lemmata. At first we show that we need only consider stopping times with finite expectation. Let us denote this set by $\mathcal{S}_1$.

3.1.2 Lemma: *If condition (L1) is fulfilled, then the optimal value function is finite, and*

$$v(x) = \sup_{\tau \in \mathcal{S}_1} E_x \left(g(X_\tau) - c\tau \right)$$

for all $x \in E$.

Proof: Let x be an arbitrary element of E. Due to

$$g(X_t) - ct \leq \sup_{s \geq 0} (g(X_s) - (c - \varepsilon)s) - \varepsilon t$$

for all $t \geq 0$ the left-hand side tends to $-\infty$ P_x-a.s. . Furthermore for each stopping time τ due to (L1)

$$E_x \left(g(X_\tau) - c\tau \right) \leq E_x \sup_{s \geq 0} (g(X_s) - cs) < \infty \quad .$$

If $E_x \tau = \infty$ then

$$E_x \left(g(X_\tau) - c\tau \right) \leq E_x \sup_{s \geq 0} (g(X_s) - (c - \varepsilon)s) - \varepsilon E_x \tau = -\infty \quad .$$

Thus we only have to maximize among stopping times with finite expectation and the lemma is proved. □

Thus only stopping times with finite expectation are of interest, and this permits an application of the following lemma.

3.1.3 Lemma: *For each stopping time τ with $E_x \tau < \infty$*

$$E_x \, u(X_\tau) \leq u(x) + E_x \, \tau \tag{3.5}$$

for all $x \in E$.

Proof: Since u is a non-negative solution of $Au = 1$, $(u(X_t) - t)_{t\geq 0}$ defines a local martingale. We use the sequence of reducing stopping times τ_n,

$$\tau_n = \inf\{t \geq 0 : |X_t - m| \geq r_n\} \quad , \tag{3.6}$$

with r_n an increasing sequence in $(0, l)$ converging to l. Since the diffusion is non-exploding, τ_n increases to infinity. Furthermore,

$$E_x\, u(X_{\tau\wedge\tau_n}) = u(x) + E_x\, \tau \wedge \tau_n \tag{3.7}$$

for all $x \in (m - r_n, m + r_n)$, since u is bounded there. The right-hand side of (3.7) increases to $E_x\, \tau$ and with Fatou's lemma we get

$$E_x\, u(X_\tau) \leq \liminf_{n\to\infty} E_x\, u(X_{\tau\wedge\tau_n}) = u(x) + E_x\, \tau \quad .$$

Hence the assertion holds . $\square$

We are now prepared for a proof of Theorem 3.1.1 . Due to (L2), the function $g(x) - cu(x)$ is bounded with maximum attained at $m \pm \psi(c)$. We start by showing that the first exit time τ^* from $(m-\psi(c), m+\psi(c))$ is optimal for starting points $x \in (m - \psi(c), m + \psi(c))$. For each stopping time τ with $E_x\tau < \infty$, due to Lemma 3.1.3 and (L2),

$$\begin{aligned} E_x\left(g(X_\tau) - c\tau\right) &= E_x(g(X_\tau) - cu(X_\tau) + cu(X_\tau) - c\tau) \\ &\leq (g - cu)(m + \psi(c)) + cu(x) \quad . \end{aligned}$$

The upper bound on the right-hand side is attained by τ^*, since

$$u(m + \psi(c)) = E_x\, u(X_{\tau^*}) = u(x) + E_x\tau^* \quad , \tag{3.8}$$

see Proposition 2.1.2. Hence τ^* is optimal, and each $x \in (m - \psi(c), m + \psi(c))$ is contained in the continuation region. Furthermore, the optimal value fulfills

$$v(x) = (g - cu)(m + \psi(c)) + cu(x) \quad .$$

Secondly, we show that immediate stopping is optimal when starting from x with $|x - m| > \psi(c)$. We use the sequence (τ_n) of reducing stopping times introduced in the proof of Lemma 3.1.3, and consider an arbitrary stopping time τ with $E_x\tau < \infty$. Due to (L1) and

$$g(X_{\tau\wedge\tau_n}) - c(\tau \wedge \tau_n) \geq \inf_{z\in E} g(z) - c\tau \quad \text{for all } n \in I\!N \quad ,$$

we may apply the dominated convergence theorem and obtain

$$E_x\left(g(X_\tau) - c\tau\right) = \lim_{n\to\infty} E_x(g(X_{\tau\wedge\tau_n}) - c(\tau\wedge\tau_n)) \quad . \tag{3.9}$$

From symmetry, we may assume w.l.o.g. $x > m + \psi(c)$. We introduce the first hitting time of $m + \psi(c)$ by

$$\sigma = \inf\{t \geq 0 : X_t = m + \psi(c)\} \quad .$$

Since u is bounded on $[m + \psi(c), m + r_n]$

$$E_x\, u(X_{\tau\wedge\tau_n\wedge\sigma}) = u(x) + E_x(\tau\wedge\tau_n\wedge\sigma) \quad , \tag{3.10}$$

and furthermore the expected payoff of $\tau\wedge\tau_n$ can be improved by $\tau\wedge\tau_n\wedge\sigma$, since σ stops at a maximal point of $g - cu$:

$$\begin{aligned} E_x\left(g(X_{\tau\wedge\tau_n}) - c(\tau\wedge\tau_n)\right) &\leq E_x\left(g(X_{\tau\wedge\tau_n}) - cu(X_{\tau\wedge\tau_n})\right) + cu(x) \\ &\leq E_x\left(g(X_{\tau\wedge\tau_n\wedge\sigma}) - cu(X_{\tau\wedge\tau_n\wedge\sigma})\right) + cu(x) \\ &= E_x(g(X_{\tau\wedge\tau_n\wedge\sigma}) - c(\tau\wedge\tau_n\wedge\sigma)) \quad . \end{aligned}$$

Boundedness of g and $Ag \leq c$ on $[m + \psi(c), m + r_n]$ imply due to (2.1.2)

$$E_x\, g(X_{\tau\wedge\tau_n\wedge\sigma}) = g(x) + E_x \int_0^{\tau\wedge\tau_n\wedge\sigma} Ag(X_s)\, ds \leq g(x) + cE_x\left(\tau\wedge\tau_n\wedge\sigma\right) \quad .$$

Hence $E_x\, g(X_{\tau\wedge\tau_n}) - c(\tau\wedge\tau_n) \leq g(x)$ and, plugging this into (3.9), we obtain $v(x) \leq g(x)$. Thus $(m + \psi(c), m + l)$ is contained in the stopping region. By symmetry, this is also true for $(m - l, m - \psi(c))$ and Theorem 3.1.1 is proved.

3.2 General applications

We want to apply the forgoing result for linear costs to two typs of reward functions. At first we consider the case, when the reward is a power of the symmetric solution u of $Au = 1$.

3.2.1 Theorem: *For the reward function $g(x) = u(x)^\alpha$ with $0 < \alpha < 1$, the conditions (L2),(L3) are valid for each cost rate $c > 0$. If additionally (L1) holds, the continuation region is given by*

$$\mathcal{C} = (m - \psi(c), m + \psi(c))$$

with $\psi(c) = \phi((\frac{c}{\alpha})^{\frac{1}{\alpha-1}})$ and ϕ denoting the inverse of $U(y) = u(m + y)$.

Proof: Note that $g(x) = G(|m-x|)$ with $G(y) = U(y)^\alpha$ for all $y \in (0,l)$. Then $G'(y)/U'(y) = \alpha U(y)^{\alpha-1}$ is decreasing, since $\alpha < 1$, with

$$\lim_{y\to 0} \frac{G'(y)}{U'(y)} = \infty \quad , \quad \lim_{y\to l} \frac{G'(y)}{U'(y)} = 0 \quad . \tag{3.11}$$

Hence the equation

$$G'(y) - cU'(y) = 0$$

has a unique solution $\psi(c)$, and $y \to G(y) - cU(y)$ attains its maximum there. Furthermore, due to $G'(y) = \alpha U(y)^{\alpha-1}U'(y)$, the above equation is equivalent to

$$U(y) = (\frac{c}{\alpha})^{\frac{1}{\alpha-1}} \tag{3.12}$$

and therefore $\psi(c) = \phi((\frac{c}{\alpha})^{1/(\alpha-1)})$.

To verify condition (L3), we compute on $(m, m+l)$

$$\begin{aligned} Ag(x) &= \frac{1}{2}a^2(x)G''(x-m) + b(x)G'(x-m) \\ &= \alpha u(x)^{\alpha-1} + \frac{1}{2}a^2(x)\alpha(\alpha-1)u(x)^{\alpha-2}u'(x)^2 \quad . \end{aligned}$$

For $x > m + \psi(c)$

$$\alpha u(x)^{\alpha-1} \leq \alpha u(m+\psi(c))^{\alpha-1} = c \quad ,$$

and therefore

$$Ag(x) \leq c \quad ,$$

since $a^2(x)\alpha(\alpha-1)u(x)^{\alpha-2}u'(x)^2 \leq 0$. Hence (L3) holds, and an application of Theorem 3.1.1 yields the assertion. □

A second application can be given for reward functions of the form $g(x) = G(|x-m|)$ with concave increasing G. Here we distinguish between the case of bounded and unbounded E and consider mean reverting diffusions, where $b(m+y) \leq 0$ for all $y \in (0,l)$.

3.2.2 Theorem: *Let $l < \infty$, and let X be a mean reverting diffusion on $E = (m-l, m+l)$. Let $g(x) = G(|m-x|)$ be a reward function with strictly increasing , concave C^2-function G. Then the continuation region is given by*

$$\mathcal{C} = (m - \psi(c), m + \psi(c)) \quad ,$$

where $\psi(c)$ is the unique solution of the equation

$$G'(y) = cU'(y) \quad \text{on} \;\; (0,l).$$

Furthermore:

(i) If for some $\alpha_1, q_1, p_1 > 0, \gamma_1 \geq 0$

$$\lim_{y\to l} U'(y)(l-y)^{\alpha_1} = p_1 \quad , \quad \lim_{y\to l} \frac{G'(y)}{q_1(l-y)^{\gamma_1}} = 1 \quad , \tag{3.13}$$

then

$$\lim_{c\to 0}(l-\psi(c))(\frac{cp_1}{q_1})^{-\frac{1}{\alpha_1+\gamma_1}} = 1 \quad . \tag{3.14}$$

(ii) If for some $\alpha_2, q_2, p_2 > 0, \gamma_2 \geq 0$

$$\lim_{y\to 0} \frac{U'(y)}{y^{\alpha_2}} = p_2 \quad , \quad \lim_{y\to 0} G'(y) y^{\gamma_2} = q_2 \quad , \tag{3.15}$$

then

$$\lim_{c\to\infty} \psi(c)(\frac{cp_2}{q_2})^{\frac{1}{\alpha_2+\gamma_2}} = 1 \quad . \tag{3.16}$$

Proof: We examine the conditions (L1)-(L3) and then apply Theorem 3.1.1. Since G is concave, it is bounded on $(0,l)$ and condition (L1) is satisfied. Since X is mean reverting, the even solution u of $Au = 1$ is convex, see 2.1.4. Thus G'/U' is decreasing with

$$\lim_{y\to l} \frac{G'(y)}{U'(y)} = 0 \quad , \quad \lim_{y\to 0} \frac{G'(y)}{U'(y)} = \infty \quad . \tag{3.17}$$

From this it follows that $y \to G(y) - cU(y)$ has a unique maximum attained at a point $\psi(c)$ with ψ denoting the inverse function of $F = G'/U'$.

Finally condition (L3) holds true, since $b(x) \leq 0$ on $(m, m+l)$, and therefore

$$Ag(x) = \frac{1}{2}a^2(x)g''(x) + b(x)g'(x) \leq 0 \quad .$$

Thus Theorem 3.1.1 provides that the continuation region has the form

$$\mathcal{C} = (m - \psi(c), m + \psi(c)) \quad .$$

Due to

$$\lim_{y \to l} \frac{G'(y)}{U'(y)} (l-y)^{-(\alpha_1+\gamma_1)} = \frac{q_1}{p_1} \quad , \quad \lim_{y \to 0} \frac{G'(y)}{U'(y)} y^{\alpha_2+\gamma_2} = \frac{q_2}{p_2} \tag{3.18}$$

we have

$$\lim_{c \to 0} c\,(l - \psi(c))^{-(\alpha_1+\gamma_1)} = \frac{q_1}{p_1} \quad , \quad \lim_{c \to \infty} c\,\psi(c)^{\alpha_2+\gamma_2} = \frac{q_2}{p_2} \quad , \tag{3.19}$$

from which the assertion follows. □

Almost the same result holds true for unbounded E. Only condition $(L1)$ has to be assumed separately. We obtain

3.2.3 Theorem: *Let X be a mean reverting diffusion on $E = \mathbb{R}$. We assume that condition (L1) holds for a reward function $g(x) = G(|m - x|)$ with strictly increasing , concave C^2-function G. Then the continuation region coincides with*

$$\mathcal{C} = (m - \psi(c), m + \psi(c)) \quad ,$$

where $\psi(c)$ denotes the unique solution of the equation

$$G'(y) = cU'(y) \quad \text{on} \quad (0, \infty).$$

Furthermore:

(i) If for some $\alpha_1, p_1, q_1 > 0, 0 \le \gamma_1 < 1$

$$\lim_{y \to \infty} \frac{U'(y)}{y^{\alpha_1}} = p_1 \quad , \quad \lim_{y \to \infty} G'(y) y^{\gamma_1} = q_1 \quad , \tag{3.20}$$

then

$$\lim_{c \to 0} \psi(c) (\frac{p_1}{q_1} c)^{\frac{1}{\alpha_1+\gamma_1}} = 1 \quad . \tag{3.21}$$

(ii) If for some $\alpha_2, p_2, q_2 > 0, \gamma_2 \ge 0$

$$\lim_{y \to 0} \frac{U'(y)}{y^{\alpha_2}} = p_2 \quad , \quad \lim_{y \to 0} G'(y) y^{\gamma_2} = q_2 \quad , \tag{3.22}$$

then

$$\lim_{c \to \infty} \psi(c) (\frac{p_2}{q_2} c)^{\frac{1}{\alpha_2+\gamma_2}} = 1 \quad . \tag{3.23}$$

3.3 Special applications

We want to apply the preceding results to the three main examples introduced in Chapter 2. We investigate for several reward functions how the boundary of the continuation region depends asymptotically on the cost constant c. This will be useful in the analysis of the stopping problem with nonlinear costs in the following chapters. At first we consider Brownian motion.

3.3.1 Brownian motion

The differential generator of Brownian motion is given by

$$A = \frac{1}{2}\partial_x^2 \quad .$$

We consider the origin as midpoint, and note that $u(x) = x^2$ is the even solution of $Au = 1$ vanishing at zero. Theorem 3.2.3 for Brownian motion takes the following form:

3.3.2 Theorem: *Let $g(x) = G(|x|)$ be a reward function with strictly increasing, concave C^2-function G. If*

$$\lim_{y\to\infty} G'(y)y^{\gamma_1} = q_1 \quad \textit{with} \;\; q_1 > 0, 0 \leq \gamma_1 < 1,$$

then the continuation region satisfies $\mathcal{C} = (-\psi(c), \psi(c))$ with

$$\psi(c) = (\frac{2}{q_1}c)^{-\frac{1}{1+\gamma_1}}(1 + o(1)) \quad \textit{for } c \to 0. \tag{3.24}$$

If $\lim_{y\to 0} G'(y)y^{\gamma_2} = q_2$ for some $\gamma_2 \geq 0, q_2 > 0$, then $\mathcal{C} = (-\psi(c), \psi(c))$ with

$$\psi(c) = (\frac{2}{q_2}c)^{-\frac{1}{1+\gamma_2}}(1 + o(1)) \quad \textit{for } c \to \infty. \tag{3.25}$$

Proof: The assertion follows immediately from Theorem 3.2.3, since $U'(y) = 2y$. □

The preceding result derives the asymptotic behaviour of the boundary $\psi(c)$ of the continuation region for c tending to zero, and infinity respectively. For special reward functions we can determine $\psi(c)$ explicitly.

1. $g(x) = |x|$:
   This reward is strongly related to sequential statistics, see 2.3.1 . We obtain
$$\psi(c) = \frac{1}{2c} \qquad \text{for all } c > 0 \quad . \tag{3.26}$$

2. $g(x) = |x|^\alpha$ with $0 < \alpha \leq 1$:
   This is an example of a concave reward function. We get the boundary
$$\psi(c) = (\frac{2}{\alpha}c)^{-\frac{1}{2-\alpha}} \qquad \text{for all } c > 0 \quad , \tag{3.27}$$
   by solving $\alpha y^{\alpha-1} - 2cy = 0$.

3. $g(x) = |x|^\alpha$ with $1 < \alpha < 2$:
   This is a convex reward and Theorem 3.2.3 yields that the continuation region is the interval $(-\psi(c), \psi(c))$ with
$$\psi(c) = (\frac{2}{\alpha}c)^{-\frac{1}{2-\alpha}} \qquad \text{for all } c > 0.$$
   Although this result coincides with the preceding one, we have seperated it since the arguments differ.

4. $g(x) = \log(1 + |x|)$:
   Again this is a concave reward and Theorem 3.2.3 yields that the continuation region is of the form $(-\psi(c), \psi(c))$. The boundary
$$\psi(c) = -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{2c}} \tag{3.28}$$
   is obtained by solving $(1 + y)2y = \frac{1}{c}$.

3.3.3 Process of posterior probabilities

We continue the example introduced in 2.2.2. Thus we consider an A-diffusion with generator
$$A = \frac{1}{2}x^2(1-x)^2\partial_x^2 \tag{3.29}$$
on $E = (0, 1)$. Using the midpoint $m = \frac{1}{2}$, a symmetric solution u of $Au = 1$, vanishing at m, is given by
$$u(x) = 2(2x - 1)\log\frac{x}{1-x} \qquad \text{for all } x \in (0, 1).$$

It is of the form $U(|x-\frac{1}{2}|)$ with $U:(0,1/2)\to(0,\infty)$ satisfying

$$U(y) = 4y\log\frac{1/2+y}{1/2-y} \quad , \quad U'(y) = 4\log\frac{1/2+y}{1/2-y} + \frac{4y}{(1/2+y)(1/2-y)}.$$

Hence

$$\lim_{y\to 1/2} U'(y)(1/2-y) = 2 \quad , \quad \lim_{y\to 0}\frac{U'(y)}{y} = 32 \quad , \tag{3.30}$$

and Theorem 3.2.2 takes the form:

3.3.4 Theorem: *Let $g(x)=G(|x-1/2|)$, $x\in(0,1)$, be a reward function with strictly increasing concave C^2-function G. Then the continuation region satisfies $\mathcal{C}=(-\psi(c),\psi(c))$ and*

$$(1/2-\psi(c)) = (\frac{2c}{q_1})^{\frac{1}{1+\gamma_1}}(1+o(1)) \qquad \text{for } c\to 0 \quad , \tag{3.31}$$

if $\lim_{y\to 1/2} G'(y)(1/2-y)^{-\gamma_1} = q_1$ for some $q_1>0, \gamma_1\geq 0$.
If $\lim_{y\to 0} G'(y)y^{\gamma_2} = q_2$ for some $q_2>0,\gamma_2\geq 0$, then

$$\psi(c) = (\frac{32c}{q_2})^{-\frac{1}{1+\gamma_2}}(1+o(1) \qquad \text{for } c\to\infty \quad . \tag{3.32}$$

Proof: The assertion is an immediate consequence of Theorem 3.2.2 together with (3.30). □

We consider the following examples for reward functions.

1. $g(x)=|x-1/2|$:
   This reward was introduced in 2.3.2 yielding an optimal Bayes test for simple hypothesis for the drift of a Brownian motion. An application of the preceding theorem shows that the continuation region fulfills $\mathcal{C}=(1/2-\psi(c),1/2+\psi(c))$ where $\psi(c)$ is the unique solution of

   $$1-c\Big(4\log\frac{1/2+y}{1/2-y}+\frac{4y}{(1/2+y)(1/2-y)}\Big)=0 \quad \text{on } (0,1/2).$$

   Furthermore we obtain

   $$\begin{aligned}\frac{1}{2}-\psi(c) &= 2c(1+o(1)) \qquad \text{for } c\to 0,\\ \psi(c) &= \frac{1}{32c}(1+o(1)) \qquad \text{for } c\to\infty.\end{aligned} \tag{3.33}$$

2. $g(x) = G(|x - \frac{1}{2}|)$ with $G(y) = -(\frac{1}{2} - y)^\alpha$ for $\alpha \geq 1$:
   This is a further example for a concave reward and Theorem 3.3.4 shows $\mathcal{C} = (\frac{1}{2} - \psi(c), \frac{1}{2} + \psi(c))$ with

$$\begin{aligned} \frac{1}{2} - \psi(c) &= (2c/\alpha)^{\frac{1}{\alpha}}(1 + o(1)) \quad \text{for } c \to 0, \\ \psi(c) &= \frac{q}{32c}(1 + o(1)) \quad \text{for } c \to \infty \quad , \end{aligned} \tag{3.34}$$

   with $q = \alpha(\frac{1}{2})^{\alpha-1}$.

3. $g(x) = u(x)^\alpha = \left(2(2x-1)\log\frac{x}{1-x}\right)^\alpha$ with $0 < \alpha < 1$:
   This is an example for an unbounded reward function, where we can apply Theorem 3.2.1. Thus the continuation region fulfills $\mathcal{C} = (\frac{1}{2} - \psi(c), \frac{1}{2} + \psi(c))$, and $\psi(c)$ is the unique solution of

$$(\frac{c}{\alpha})^{\frac{1}{\alpha-1}} = U(y) = 4y \log\frac{\frac{1}{2}+y}{\frac{1}{2}-y}$$

   on $(0, \frac{1}{2})$.

3.3.5 Portfolio optimization

We consider portfolio strategies without transaction costs that initially invest a fraction of capital in a risky asset and hold this until stopping. The fraction of wealth in the risky asset is a diffusion with generator

$$A = \frac{1}{2}x^2(1-x)^2\partial_x^2 + x(1-x)(\frac{1}{2} - x)\partial_x \quad , \tag{3.35}$$

see 2.2.3 and 2.3.3. This is symmetric w.r.t. the midpoint $m = \frac{1}{2}$ and mean reverting. Furthermore a symmetric solution of $Au = 1$ vanishing at m is given by

$$u(x) = (\log\frac{x}{1-x})^2 = U(|x - \frac{1}{2}|)$$

for all $x \in (0, 1)$ with

$$U(y) = (\log\frac{1/2+y}{1/2-y})^2 \quad , \quad U'(y) = 2\frac{\log\frac{1/2+y}{1/2-y}}{(1/2+y)(1/2-y)}$$

for all $y \in (0, \frac{1}{2})$. We consider some reward functions

1. $g(x) = u(x)^\alpha = |\log(x/(1-x))|^{2\alpha}$ with $0 < \alpha < 1$:
   Theorem 3.2.1 and Corollary 2.2.5 yield
$$\mathcal{C} = (\frac{1}{2} - \psi(c), \frac{1}{2} + \psi(c)) \quad ,$$
   with $\psi(c)$ the unique solution of
$$(\log \frac{1/2+y}{1/2-y})^2 = (\frac{c}{\alpha})^{\frac{1}{\alpha-1}} \quad \text{on} \quad (0, \frac{1}{2}).$$
   Thus
$$\psi(c) = \frac{1}{2} \frac{\exp((\frac{c}{\alpha})^{\frac{1}{2\alpha-2}}) - 1}{\exp((\frac{c}{\alpha})^{\frac{1}{2\alpha-2}}) + 1} \qquad \text{for all } c > 0. \tag{3.36}$$

2. $g(x) = |\log \frac{x}{1-x}|$:
   This is a special case of the preceding example with $\alpha = \frac{1}{2}$. Thus $\mathcal{C} = (\frac{1}{2} - \psi(c), \frac{1}{2} + \psi(c))$ with
$$\psi(c) = \frac{1}{2} \frac{\exp(\frac{1}{2c}) - 1}{\exp(\frac{1}{2c}) + 1} \quad .$$

3. $g(x) = \begin{cases} \log \frac{1}{1-x} & \text{,if } x > \frac{1}{2} \\ \log \frac{1}{x} & \text{,if } x < \frac{1}{2} \end{cases}$:
   Here we want to apply Theorem 3.1.1 and have to verify (L1)-(L3). The first condition holds as in the second example. To prove (L2) note that $G(y) = g(\frac{1}{2} + y)$ satisfies
$$G(y) = -\log(\frac{1}{2} - y) \quad , \quad G'(y) = \frac{1}{1/2 - y} \quad . \tag{3.37}$$
   Thus
$$F(y) = \frac{G'(y)}{U'(y)} = \frac{1}{2} \frac{1/2+y}{\log \frac{1/2+y}{1/2-y}}$$
   is strictly decreasing with $\lim_{y\to 0} F(y) = \infty$, $\lim_{y\to 1/2} F(y) = 0$. Hence $y \to G(y) - cU(y)$ has a unique maximum attained at $\psi(c)$, and $\psi(c)$ is the unique solution of $F(y) = c$ on $(0, \frac{1}{2})$. The condition (L3) follows since
$$Ag(x) = \frac{1}{2} x(1-x)$$
   is decreasing on $(\frac{1}{2}, 1)$. Therefore Theorem 3.1.1 shows that the continuation region fulfills $\mathcal{C} = (\frac{1}{2} - \psi(c), \frac{1}{2} + \psi(c))$ for all $c > 0$.

Chapter 4

Concave costs of observations

4.1 Asymptotics of the continuation region

In the case of nonlinear costs, the analysis of the corresponding optimal stopping problem becomes more difficult. As before we consider an A-diffusion X on an open interval $E = (m-l, m+l)$. We assume that X is symmetric w.r.t. the midpoint m, see Definition 2.1.3. Instead of linear costs we investigate payoff functions of the form

$$(x,t) \to g(x) - c(t)$$

with concave increasing c and continuous, w.r.t. m symmetric g.

We recall Chapter 2 and note that, starting from $x \in E$ at time t, the optimal value is denoted by

$$v(x,t) = \sup_{\tau \in \mathcal{S}} E_x \left(g(X_\tau) - c(t+\tau) \right) \quad .$$

for all $x \in E$, $t \geq 0$. The continuation region $\mathcal{C}$ is defined by

$$\mathcal{C} = \{(x,t) : v(x,t) > g(x) - c(t)\}$$

and its compliment is called stopping or early exercise region.
For concave cost functions the cost rate decreases. Thus one may expect that the continuation region increases. This is indeed true and we will determine in this section its asymptotic growth rate. We assume the following conditions for the cost function $c : [0,\infty) \to [0,\infty)$.

(cc1) c is strictly increasing with $\lim_{t\to\infty} c(t) = \infty$,

(cc2) c is twice continuously differentiable and concave ,

(cc3) For each $x \in E$ there exists an $\alpha \in (0,1)$ such that

$$E_x \sup_{t\geq 0}(g(X_t) - \alpha c(t)) < \infty \quad .$$

Since c is increasing and concave the condition (cc3) implies that for each x in E there exists an $\alpha \in (0,1)$ such that for all $t_0 \geq 0$

$$E_x \sup_{t\geq 0}(g(X_t) - \alpha c(t_0 + t)) < \infty \quad . \tag{4.1}$$

Hence, as in the linear case,

$$\lim_{t\to\infty}(g(X_t) - c(t_0 + t)) = -\infty \quad P_x \text{ a.s.} \quad \text{for all } x \in E, t_0 \geq 0. \tag{4.2}$$

Furthermore, for determining $v(x, t_0)$ we need only consider stopping times τ with $E_x c(t_0 + \tau) < \infty$. Condition (4.1) allows us to apply the standard results of optimal stopping for continuous time Markov processes. We obtain

4.1.1 Theorem: *If the conditions (cc1)-(cc3) are fulfilled with a symmetric continuous reward function g, then*

(i) For each $t_0 \geq 0$ the first exit time from $\mathcal{C}$,

$$\tau^*_{t_0} = \inf\{t \geq 0 : (X_t, t_0 + t) \notin \mathcal{C}\} \quad ,$$

is an optimal stopping time, satisfying $P_x(\tau^*_{t_0} < \infty) = 1$ for all $x \in E$.

(ii) The optimal value function v is lower semi-continuous on $\mathbb{R} \times [0,\infty)$, the continuation region $\mathcal{C}$ is an open subset of $\mathbb{R} \times [0,\infty)$.

(iii) If additionally the coefficients a, b of A are locally Hölder-continuous, then v is twice continously differentiable in x, once continuously differentiable in t on the continuation region $\mathcal{C}$, and fulfills

$$(\partial_t + A)v = 0 \quad \text{on } \mathcal{C} \quad .$$

Proof: Assertions (i) follows from Shiryayev [56], Theorem 6, with following corollary. Approximating v by the optimal value function of the truncated stopping problem with finite horizon yields an increasing sequence of lower semi-continuous functions. Hence the limit v is lower semi-continuous itsself, see section 3.2.4 Shiryayev [56]. Furthermore this implies that $\mathcal{C}$ is an open subset of $E \times [0, \infty)$.

It is well known, that the optimal value function is harmonic on the continuation region. In the case of a one-dimensional non-exploding diffusion this implies that v is continuous on G, see Lai [39], Theorem 2 and the remark on page 423. To examine that v fulfills the above parabolic equation, we fix $(x_0, t_0) \in \mathcal{C}$ and consider an open rectangle $R = (x_1, x_2) \times (t_1, t_2)$ contained in $\mathcal{C}$ such that $(x_0, t_0) \in R$. The first-initial boundary value problem

$$\begin{aligned} (\partial_t + A)w &= 0 \qquad \text{on } R \\ w &= v \qquad \text{on } \partial R \end{aligned} \tag{4.3}$$

with $\partial R = [x_1, x_2] \times \{t_2\} \cup \{x_1\} \times [t_1, t_2] \cup \{x_2\} \times [t_1, t_2]$ has a unique solution w, see Friedman [22], Theorem 3.6, page 138. Since v is harmonic, it coincides with w on R and is therefore a solution of the above partial differential equation, due to

$$\begin{aligned} v(x,t) &= E_x v(X_\tau, t+\tau) = E_x w(X_\tau, t+\tau) \\ &= w(x,t) + E_x \int_0^\tau (\partial_t + A) w(X_s, t+s)ds = w(x,t) \end{aligned} \tag{4.4}$$

for all $(x,t) \in R$ with τ denoting the first exit time from R. □

The preceding theorem in principle gives the solution to the optimal stopping problem. But it remains to determine the continuation region explicitly. As was pointed out in the introduction this seems not to be possible. Thus we want to give in the following an asymptotic description of the continuation region. As a first step we will determine some facts on its shape with the following lemma. Let us denote the difference between optimal expected and immediate payoff by

$$d(x,t) = v(x,t) - (g(x,t) - c(t)) \quad .$$

4.1.2 Lemma: *If the conditions (cc1)-(cc3) are fulfilled with a symmetric continuous reward function g, then*

(i) $v(x,\cdot)$ *is decreasing for all* $x \in E$,

(ii) $d(x,\cdot)$ *is increasing for all* $x \in E$,

(iii) $v(\cdot,t)$ *is even w.r.t.* m *for all* $t \geq 0$, *and the continuation region is symmetric w.r.t.* m, *i.e.*

$$(m+y,t) \in \mathcal{C} \Longleftrightarrow (m-y,t) \in \mathcal{C} \quad \textit{for all } y \in (0,l), t \geq 0 \quad .$$

Proof: Since c is increasing, the expected payoff $E_x\left(g(X_\tau)-c(t+\tau)\right)$ decreases in t for each stopping time τ. Thus assertion (i) follows. To verify (ii) we note that $c(t+\tau)-c(t)$ is decreasing in t as c is concave. Hence,

$$d(x,t) = \sup_\tau E_x\left(g(X_\tau) - g(x) - (c(t+\tau) - c(t))\right)$$

is increasing in t . (iii) follows from the fact that the law of X w.r.t. P_x coincides with the law of $2m-X$ w.r.t. P_{2m-x}, see Chapter 2.1.5, and from $g(x) = g(2m-x)$ for all $x \in E$. □

We want to apply the results for linear costs to obtain an inner approximation of the continuation region, and to achieve this, we make the following assumptions for the reward function g. Recall that $G(y) = g(m+y), U(y) = u(m+y)$ for all $y \in (0,l)$.

(R1) G is twice continuously differentiable and strictly increasing.

(R2) $\frac{G'}{U'}$ is strictly decreasing with

$$\lim_{y\to 0} \frac{G'(y)}{U'(y)} = \infty \quad , \quad \lim_{y \to l} \frac{G'(y)}{U'(y)} = 0 \quad .$$

(R3) Ag is decreasing on $(m, m+l)$ or $Ag \leq 0$.

Although not explicitly mentioned, these conditions were often examined in the examples of Chapter 3. In the following we show that mean reverting diffusions satisfy (R1)-(R3) for concave G.

4.1.3 Proposition *Let $g(x) = G(|x-m|)$ be a reward with strictly increasing concave C^2-function G. Then the conditions (R1)-(R3) are valid for any mean reverting diffusion.*

Proof: We have to verify (R2) and (R3). We recall that U is strictly convex in the case of a mean reverting diffusion, see 2.1.4. Hence U' is strictly increasing. Furthermore, due to concavity the derivative G' is decreasing. Hence $F = G'/U'$ is strictly decreasing. Furthermore, $\lim_{y\to l} F(y) = 0$ and $\lim_{y\to 0} F(y) = +\infty$, since $U'(0) = 0$ and $U'(l) = +\infty$. Thus (R2) is valid. Condition (R3) holds since X is mean reverting and therefore $Ag(x) = \frac{1}{2}a^2(x)g''(x) + b(x)g'(x) \leq 0$ for all $x > m$. $\square$

In the following we define an increasing curve β_- which will lead to an inner approximation of the continuation region for concave costs of observations.

4.1.4 Definition: *We define the function $F : (0,l) \to (0,\infty)$ by*

$$F(y) = \frac{G'(y)}{U'(y)} \quad \textit{for all } y \in (0,l) \quad . \tag{4.5}$$

If condition (R2) is valid, F has an inverse function which we denote by ψ. Furthermore, for a concave cost function c we define the increasing curve β_- by

$$\beta_-(t) = \psi(c'(t)) \quad \textit{for all } t > 0. \tag{4.6}$$

Thus the moving boundary β_- is determined by applying the results for linear costs to each cost rate $c'(t)$. This is possible as will be explained below.

The C^1-function ψ is strictly decreasing and a one-to-one map of $(0,\infty)$ onto $(0,l)$. The point $\psi(k)$ is the unique solution of the equation

$$G'(y) = kU'(y) \tag{4.7}$$

on $(0,l)$ for any $k \in (0,\infty)$. Thus the function

$$y \to G(y) - kU(y)$$

has $\psi(k)$ as unique extremal point in $(0,l)$ which must be a maximum due to (R2). Thus condition (L2) holds true and furthermore

$$G''(\psi(k)) \leq kU''(\psi(k)) \quad , \quad G'(\psi(k)) = kU'(\psi(k)) \quad . \tag{4.8}$$

Hence,

$$Ag(m+\psi(k)) \leq kAu(m+\psi(k)) = k \quad . \tag{4.9}$$

From Ag is decreasing on $(m, m+l)$ it follows $Ag(x) \leq k$ for all $|x-m| \geq \psi(k)$. Thus we have verified that condition (L3) holds, and we can state the following lemma.

4.1.5 Lemma: *If g satisfies (R1)-(R3) and c fulfills (cc1)-(cc3), then the conditions (L1)-(L3) are valid for each cost rate $k > 0$. Thus the results for linear costs can be used.*

As a first application we derive the following inner approximation.

4.1.6 Proposition *Under the conditions (cc1)-(cc3), (R1)-(R3), the set*

$$\mathcal{C}_{in} = \{(x,t) : |x-m| < \psi(c'(t))\}$$

is contained in the continuation region

Proof: Let us fix an arbitrary $t_0 > 0$. Since the cost function c is concave, $c(t_0+t) - c(t) \leq c'(t_0)t$ for all $t \geq 0$. An application of the linear case, see Theorem 3.1.1, yields for $|x-m| < \psi(c'(t_0))$ the existence of a stopping time τ such that

$$E_x\left(g(X_\tau) - c'(t_0)\tau\right) > g(x) \quad .$$

Hence

$$\begin{aligned} v(x,t_0) &\geq E_x\left(g(X_\tau) - c(t_0+\tau)\right) \geq E_x(g(X_\tau) - c'(t_0)\tau) - c(t_0) \\ &> g(x) - c(t_0) \quad . \end{aligned}$$

and the assertion is proved. □

We can use the inner approximation to prove that the continuation region is a set enscribed between two boundary curves $m \pm \beta^*(t)$. This basically relies on

$$Ag(x) \leq c'(t)$$

for all (x,t) with $|x-m| > \psi(c'(t))$.

4.1.7 Theorem: *If the cost function c fulfills (cc1)-(cc3) and the reward function g satisfies (R1)-(R3), then there exists an increasing function β^* such that*

$$\mathcal{C} = \{(x,t) : |x-m| < \beta^*(t)\} \quad .$$

The boundary function β^* is continuous from the left and fulfills

$$\psi(c'(t)) \le \beta^*(t) < l \qquad \text{for all } t > 0. \tag{4.10}$$

Proof: We define for all $t \ge 0$

$$\beta^*(t) = \inf\{y \in (0,l) : d(m+y,t) = 0\} = \inf\{y \in (0,l) : (m+y,t) \notin \mathcal{C}\}$$

with the convention $\inf \emptyset = l$. Due to the inner approximation $\beta^*(t) \ge \psi(c'(t))$, and it remains to prove that all points (x,t_0) with $|x-m| > \beta^*(t_0)$ belong to the stopping region. From symmetry it is sufficient to consider the case $x > m + \beta^*(t_0)$. We introduce the first hitting time of the moving boundary $(m + \psi(c'(t_0+t)))_{t\ge 0}$,

$$\sigma = \inf\{t \ge 0 : X_t = m + \psi(c'(t_0+t))\} \quad , \tag{4.11}$$

and recall the optimal stopping time

$$\tau^* = \inf\{t \ge 0 : (X_t, t_0+t) \notin \mathcal{C}\} \quad .$$

Since the starting point x lies above $m + \beta^*(t_0) \notin \mathcal{C}$, the diffusion will first exit the continuation region before it reaches the moving boundary $m + \psi(c'(t_0+\cdot))$, hence $\tau^* \le \sigma$. As in the linear case, we use the sequence (τ_n) of reducing stopping times defined by

$$\tau_n = \inf\{t \ge 0 : |X_t - m| \ge r_n\}$$

with $r_n \uparrow l$, see (3.6). Dominated convergence implies

$$E_x(g(X_{\tau^*}) - c(t_0+\tau^*)) = \lim_{n\to\infty} E_x\left(g(X_{\tau^*\wedge\tau_n}) - c(t_0+\tau^*\wedge\tau_n)\right) \quad . \tag{4.12}$$

Since g is bounded on $[m-r_n, m+r_n]$,

$$E_x g(X_{\tau^*\wedge\tau_n}) = g(x) + E_x \int_0^{\tau^*\wedge\tau_n} Ag(X_s)ds \quad , \tag{4.13}$$

see 2.1.2, and

$$Ag(X_s) \leq Ag(m + \psi(c'(t_0 + s))) \leq c'(t_0 + s) \quad ,$$

due to $X_s \geq m + \psi(c'(t_0 + s))$ for all $s \leq \tau^* \leq \sigma$ and (R3). Thus

$$E_x \int_0^{\tau^* \wedge \tau_n} Ag(X_s) ds \leq E_x c(t_0 + \tau^* \wedge \tau_n) - c(t_0) \quad ,$$

and inserting into the above equation 4.13 shows

$$E_x(g(X_{\tau^*}) - c(t_0 + \tau^*)) \leq g(x) - c(t_0) \quad .$$

Hence (x, t_0) is contained in the stopping region.

The boundary β^* is increasing since the difference $d(x,t)$ between immediate payoff and optimal payoff increases in t for fixed x. A point (x,t) lies in the continuation region if $d(x,t) > 0$, but then $d(x, t+s) > 0$ and therefore $(x, t+s) \in \mathcal{C}$ for all $s \geq 0$.

The increasing function β^* cannot reach l in finite time, say t_0. Otherwise the region $E \times (t_0, \infty)$ would be contained in the continuation region and the first exit time from $\mathcal{C}$ would not be finite when starting after t_0, and this would contradict Theorem 4.1.1.

To prove the continuity from the left let (t_n) be a sequence increasing to t. Then $\beta^*(t_n)$ increases to $\beta^*(t-) \leq \beta^*(t)$. Since the stopping region $\mathcal{E}$ is closed the sequence $(\beta^*(t_n), t_n)$ converges in $\mathcal{E}$ to $(\beta^*(t-), t)$. Hence the other inequality $\beta^*(t) \leq \beta^*(t-)$ is valid, and the theorem is proved. □

We now know that the continuation region is an open set enscribed between the moving boundaries $m \pm \beta^*(t)$. In the following, we will show as a main result that the growth of the inner approximation coincides asymptotically with that of the continuation region, i.e.

$$1 = \begin{cases} \lim_{t\to\infty} \frac{B - \beta^*(t)}{B - \psi(c'(t)))} & \text{,if } B < \infty \\ \lim_{t\to\infty} \frac{\beta^*(t)}{\psi(c'(t))} & \text{,if } B = \infty \end{cases} \quad . \tag{4.14}$$

with $B = \lim_{t\to\infty} \psi(c'(t))$.

To verify this we need mild additional conditions. We recall that $\psi : (0, \infty) \to (0, l)$ denotes the inverse function of $F = G'/U'$ and $\beta_-(t) = \psi(c'(t))$ for all $t > 0$. Furthermore we set $c'(\infty) = \lim_{t\to\infty} c'(t)$, $B = \lim_{t\to\infty} \psi(c'(t))$ and formulate the following condition:

There exists an increasing differentiable function $h \geq 0$ with the following properties:

(cc4) $\lim_{x \downarrow c'(\infty)} h(x) = 0 \quad , \quad \lim_{x \downarrow c'(\infty)} x h'(x) = 0 \quad ,$

(cc5) The curve β_+ defined by

$$\beta_+(t) = \psi(c'(t)(1 - h(c'(t))) \quad \text{for all } t > 0 \tag{4.15}$$

is asymptotically equivalent to β_-, i.e.

$$1 = \begin{cases} \lim_{t\to\infty} \frac{B-\beta_-(t)}{B-\beta_+(t)} & , \text{ if } B < \infty \\ \lim_{t\to\infty} \frac{\beta_-(t)}{\beta_+(t)} & , \text{ if } B = \infty \end{cases} \quad ,$$

(cc6)

$$\lim_{t\to\infty} \frac{c''(t) U(\beta_+(t))}{h(c'(t))c'(t)} = 0 \quad .$$

Note that $m \pm \beta_-(t)$ determines the boundary of the inner approximation $\mathcal{C}_{in}$. Since ψ is decreasing, β_+ exceeds β_- and both are asymptotically equivalent in the sense of condition (cc5). In the following we will see that asymptotically the optimal boundary β^* lies between β_- and β_+. In a first step we will construct a function $\Phi : E \times (0, \infty) \to I\!R$, superharmonic for large t, that exceeds $g(x) - c(t)$ and touches it at the curves $m \pm \beta_+(t)$. The following lemma gives the precise formulation:

4.1.8 Lemma: *Let the reward function g fulfill the conditions (R1)-(R3), and let the cost function c satisfy (cc1)-(cc6). Then there exists a function*

$$\phi : E \times (0, \infty) \to I\!R$$

and some $t_0 > 0$ such that

(i) $(\partial_t + A)\phi(x,t) \leq 0$ *for all $x \in E, t \geq t_0$.*

(ii) ϕ *is even w.r.t. m .*

(iii) $\phi(x,t) \geq g(x) - c(t)$ *for all $x \in E, t > 0$.*

(iv) $\phi(m \pm \beta_+(t), t) = g(m \pm \beta_+(t)) - c(t)$ *for all $t > 0$.*

Proof: At first we recall that $x \to g(x) - ku(x)$ has a unique maximum at $m \pm \psi(k)$ for each $k \in (0, \infty)$. Furthermore, u is a non-negative symmetric function satisfying $Au = 1$. We define

$$\eta(t) = 1 - h(c'(t)) \quad , \quad f(t) = \eta(t)c'(t) \quad \text{for all } t > 0 \tag{4.16}$$

and

$$\phi_1(x,t) = \eta(t)c'(t)u(x) - c(t) \quad \text{for all } t > 0, x \in E \quad . \tag{4.17}$$

Due to $(cc4)$, η increases to 1, and f fulfills

$$f'(t) = \eta'(t)c'(t) + \eta(t)c''(t) = c''(t)(1 - h(c'(t))) - c''(t)h'(c'(t))c'(t) \quad .$$

Hence $f'(t) \leq 0$ for large t. From this it follows that ϕ_1 is superharmonic for large t and $x \in E$, since

$$\begin{aligned}(\partial_t + A)\phi_1(x,t) &= f'(t)u(x) - c'(t) + \eta(t)c'(t) \\ &= f'(t)u(x) - c'(t)(1 - \eta(t)) \leq 0 \quad .\end{aligned}$$

In the next step, ϕ_1 will be lifted up such that it exceeds $(x,t) \to g(x) - c(t)$ and touches it at the curves $m \pm \beta_+(t)$. For a fixed $t > 0$ the function

$$x \to g(x) - c'(t)\eta(t)u(x)$$

has its maximum at $m \pm \psi(c'(t)\eta(t)) = m \pm \beta_+(t)$. Thus

$$x \to c'(t)\eta(t)u(x) + \gamma(t)$$

with $\gamma(t) = g(m + \beta_+(t)) - c'(t)\eta(t)u(m + \beta_+(t))$ exceeds the function g and touches it at $x = m \pm \beta_+(t)$. Hence

$$\phi(x,t) = \phi_1(x,t) + \gamma(t) = \eta(t)c'(t)u(x) + \gamma(t) - c(t) \tag{4.18}$$

fulfills the properties (ii)-(iv).

To prove (i) we note that

$$\begin{aligned}\gamma'(t) &= g'(m + \beta_+(t))\beta_+{}'(t) \\ &\quad - u'(m + \beta_+(t))\beta_+{}'(t)c'(t)\eta(t) - u(m + \beta_+(t))f'(t) \\ &= -f'(t)u(m + \beta_+(t))\end{aligned}$$

since $g'(m+\psi(k)) - ku'(m+\psi(k)) = 0$ for all $k \in (0,\infty)$. Furthermore

$$\begin{aligned}(\partial_t + A)\phi(x,t) &= f'(t)u(x) - c'(t)(1-\eta(t)) + \gamma'(t) \\ &= f'(t)u(x) - c'(t)(1-\eta(t)) - f'(t)u(m+\beta_+(t))\end{aligned}$$

and ϕ is superharmonic for large t, if $f'(t)u(m+\beta_+(t))$ tends to zero faster than $c'(t)(1-\eta(t)) = c'(t)h(c'(t))$. Since

$$f'(t) = c''(t)(1+o(1)) \quad , \tag{4.19}$$

this follows from condition (cc6). □

Note that (cc5) implies the asymptotic equivalence of β_+ and β_-. Thus, if the region enscribed the curves $m \pm \beta_+(t)$ contains the continuation region for large t, we will have determined the desired asymptotic shape of the continuation region.

4.1.9 Theorem: *Let the reward function g fulfill the conditions (R1)-(R3), and let the cost function c satisfy (cc1)-(cc6). Then there exists some $t_0 > 0$ such that the continuation region $\mathcal{C}$ fulfills*

$$\mathcal{C} \cap (E \times (t_0, \infty)) \subset \mathcal{C}_{out}$$

with $\mathcal{C}_{out} = \{(x,t) : t > t_0, |m-x| < \beta_+(t)\}$. Thus

$$\beta^*(t) \le \beta_+(t) \quad \text{for all } t \ge t_0.$$

Proof: We consider the function ϕ from the preceding lemma and fix some $t_0 > 0$ such that ϕ satisfies the properties (i)-(iv). We note that as in the linear case, see 3.1.2, the optimal expected payoff $v(x,t)$ is obtained by maximizing $E_x\, g(X_\tau) - c(t+\tau)$ among all stopping times τ with $E_x c(t+\tau) < \infty$. As in the linear case we consider a sequence (ρ_n) of reducing stopping times defined by

$$\rho_n = \inf\{t \ge 0 : |X_t - m| \le r_n\} \wedge n \tag{4.20}$$

with $r_n \uparrow l$. Note that each ρ_n is bounded in time. Condition (cc3) and

$$g(X_{\tau\wedge\rho_n}) - c(t+\tau\wedge\rho_n) \ge g(m) - c(t+\tau) \quad \text{for all } n \in I\!N$$

allows us to apply dominated convergence which leads to

$$E_x\left(g(X_\tau) - c(t+\tau)\right) = \lim_{n\to\infty} E_x\left(g(X_{\tau\wedge\rho_n}) - c(t+\tau\wedge\rho_n)\right) \quad . \tag{4.21}$$

We will prove that the curves $(m \pm \beta_+(t), t)_{t\geq t_0}$ are contained in the stopping region. Then Theorem 4.1.7 implies $\beta^*(t) \leq \beta_+(t)$ for all $t \geq t_0$.

Therefore we fix $t \geq t_0$ and consider an arbitrary stopping time τ with $E_x c(t+\tau) < \infty$. Since ρ_n is bounded in time, $(\phi(X_s, t+s))_{0\leq s\leq\rho_n}$ is uniformly bounded. Hence, using the generator of the space time process

$$E_x\phi(X_{\tau\wedge\rho_n}, t+\tau\wedge\rho_n) = \phi(x,t) + E_x\int_0^{\tau\wedge\rho_n} (\partial_t + A)\phi(X_s, t+s)ds \leq \phi(x,t) \quad .$$

For $x = m \pm \beta_+(t)$ we obtain

$$\begin{aligned} E_x\, g(X_{\tau\wedge\rho_n}) - c(t+\tau\wedge\rho_n) &\leq E_x\phi(X_{\tau\wedge\rho_n}, t+\tau\wedge\rho_n) \\ &\leq \phi(x,t) = g(x) - c(t) \quad . \end{aligned}$$

Thus (4.21) shows that the immediate payoff $g(x) - c(t)$ cannot be improved by the expected by any stopping time, hence

$$v(x,t) \leq g(x) - c(t) \quad , \quad x = m \pm \beta_+(t) \quad .$$

Thus the curves $(m \pm \beta_+(t), t)_{t\geq t_0}$ are contained in the stopping region which proves the result. □

As a consequence, we obtain the asymptotics of β^* which determines the boundary of the continuation region, since $\beta_-(t) \leq \beta^*(t) \leq \beta_+(t)$ for large t. Recall $B = \lim_{t\to\infty}\psi(c'(t))$.

4.1.10 Corollary: *If the conditions of Theorem 4.1.9 hold, the boundary of the continuation region is asymptotically equivalent to its inner approximation, i.e.*

$$1 = \begin{cases} \lim_{t\to\infty} \frac{B-\beta^*(t)}{B-\beta_-(t)} & ,\text{if } B < \infty \\ \lim_{t\to\infty} \frac{\beta^*(t)}{\beta_-(t)} & ,\text{if } B = \infty \end{cases} \quad . \tag{4.22}$$

So far, we have proved that the continuation region is the region between the curves $m \pm \beta^*(t)$ and we have determined its asymptotic growth. It remains to examine the assumptions we have used, in particular (cc4)-(cc6). In the following we will see that these assumptions hold in various circumstances.

4.2 Applications

We want to apply the forgoing results to the three basic examples that accompany this thesis. We will determine the asymptotic growth rate of the continuation region for various reward and cost functions.

4.2.1 Brownian motion

We consider Brownian motion where the generator is $A = \frac{1}{2}\partial_x^2$, and $u(x) = x^2$ solves $Au = 1$. At first we will consider concave reward functions and state the following theorem corresponding to 3.3.2.

4.2.2 Theorem: *Let $g(x) = G(|x|)$ be a reward with strictly increasing concave C^2-function G satisfying*

$$\lim_{y\to\infty} G'(y)y^\gamma = q \qquad \text{for some} \;\; q > 0, 0 \leq \gamma < 1.$$

Let c be a cost function with $\lim_{t\to\infty} c'(t) = 0$, strictly increasing, twice continuously differentiable and concave. Furthermore we assume

$$\lim_{t\to\infty} \frac{c(t)}{t^\alpha} = r$$

for some $r \in (0,\infty]$ and $\alpha > (1-\gamma)/2$, and we suppose the existence of an increasing function $h \geq 0$ such that

$$\lim_{x\to 0} h(x) = 0, \lim_{x\to 0} xh'(x) = 0, \lim_{t\to\infty} \frac{c''(t)}{c'(t)^{1+\frac{2}{1+\gamma}} h(c'(t))} = 0 \quad . \tag{4.23}$$

Then the continuation region is the set enscribed between the curves $\pm\beta^*(t)$, i.e.

$$\mathcal{C} = \{(x,t) : |x| < \beta^*(t)\} \quad ,$$

and

$$\beta^*(t) = \psi(c'(t))(1+o(1)) = (\frac{2}{q}c'(t))^{-\frac{1}{1+\gamma}}(1+o(1)) \quad .$$

Proof: We have to examine the conditions (cc1)-(cc6) and (R1)-(R3). Then we can apply Theorem 4.1.7 and Corollary 4.1.10 to obtain the assertion.
Proposition 4.1.3 provides (R1)-(R3) for the reward function g. Since c grows

faster than t^α with $\alpha > \frac{1-\gamma}{2}$, condition (cc3) is valid. From the linear case, see Theorem 3.2.3 , we get the inner and outer approximation

$$\beta_-(t) = \psi(c'(t)) \quad , \quad \beta_+(t) = \psi(c'(t)(1 - h(c'(t))) \quad ,$$

which fulfill

$$\begin{aligned} \beta_-(t) &= (\frac{2}{q}c'(t))^{-\frac{1}{1+\gamma}}(1 + o(1)) \quad , \\ \beta_+(t) &= \left(\frac{2}{q}c'(t)(1 - h(c'(t))\right)^{-\frac{1}{1+\gamma}}(1 + o(1)) \quad , \end{aligned}$$

compare to (3.24). Hence they are asymptotically equivalent and (cc5) is true. Finally condition (cc6) follows, since $u(\beta_+(t)) = O(c'(t)^{-\frac{2}{1+\gamma}})$. □

We want to apply the above result to some reward and cost functions.

1. $g(x) = |x|$:
   This reward is strongly related to locally best tests, see 2.3.1. The inverse function of G'/U' is given by $\psi(z) = 1/(2z)$ for all $z \in (0, \infty)$. Thus the inner approximation satisfies

   $$\beta_-(t) = \frac{1}{2c'(t)} \qquad \text{for all } t > 0.$$

   We consider some cost functions

   1.1 $c(t) = t^\alpha$ with $\frac{1}{2} < \alpha < 1$:
   Then the assumptions of the preceding theorem are fulfilled, if we can find a function $h \geq 0$ satisfying (4.23). We define $h(x) = x^\delta$ with $0 < \delta < \frac{2\alpha-1}{1-\alpha}$. Then the first two properties hold. The last one follows since

   $$\lim_{t\to\infty} \frac{t^{\alpha-2}}{t^{3(\alpha-1)}t^{(\alpha-1)\delta}} = \lim_{t\to\infty} t^{-\alpha(2+\delta)+\delta+1} = 0 \quad ,$$

   the exponent being less than zero. Hence we obtain

   $$\mathcal{C} = \{(x, t) : |x| < \beta^*(t)\} \quad ,$$

   and

   $$\beta^*(t) = \frac{1}{2c'(t)}(1 + o(1)) = \frac{1}{2\alpha}t^{-(\alpha-1)}(1 + o(1)) \quad . \qquad (4.24)$$

1.2 $c(t) = t + \log(1+t)$:
Then $c'(t) = 1 + 1/(1+t)$ decreases to one and $c''(t) = -1/(1+t)^2$. Thus the continuation region remains bounded in space with limit

$$B = \lim_{t\to\infty} \psi(c'(t)) = \frac{1}{2} \quad .$$

The inner approximation is given by

$$\beta_-(t) = \frac{1}{2c'(t)} = \frac{1+t}{2(2+t)} = \frac{1}{2} - \frac{1}{2(2+t)} \quad .$$

We define $h(x) = (x-1)^\delta$ for all $x > 1$ with $1 < \delta < 2$.
Then the outer approximation is defined by

$$\beta_+(t) = \frac{1}{2c'(t)(1-h(c'(t)))} = \frac{1}{2} - \frac{(2+t)((1+t)^\delta - 1) - (1+t)^{\delta+1}}{2(2+t)((1+t)^\delta - 1)} \quad .$$

We obtain that

$$\frac{1}{2} - \beta_-(t) = \frac{1}{2(2+t)}$$

and

$$\frac{1}{2} - \beta_+(t) = \frac{1}{2(2+t)} \frac{(1+t)^\delta - 1 - (1+t)}{(1+t)^\delta - 1}$$

are asymptotically equivalent due to $\delta > 1$. Furthermore the continuation region is determined by

$$\mathcal{C} = \{(x,t) : |x| < \beta^*(t)\}$$

with

$$\frac{1}{2} - \beta^*(t) = (\frac{1}{2} - \beta_-(t))(1 + o(1)) = \frac{1}{2(2+t)}(1 + o(1)) \qquad (4.25)$$

since

$$\frac{c''(t)}{c'(t)h(c'(t))} = \frac{-1/(1+t)^2}{(1+1/(1+t))(1/(1+t))^\delta} \to 0 \quad .$$

2. $g(x) = |x|^\nu$ with $0 < \nu < 1$:
Then $F(y) = G'(y)/U'(y) = \frac{\nu}{2} y^{\nu-2}$ has the inverse function

$$\psi(z) = (\frac{2z}{\nu})^{\frac{1}{\nu-2}} \qquad \text{for all } z \in (0,\infty).$$

We consider the following cost functions

2.1 $c(t) = t^\alpha$ with $\frac{\nu}{2} < \alpha < 1$:
We apply Theorem 4.2.2 with $\gamma = 1 - \nu$ and set $h(x) = x^\delta$ with $0 < \delta < \frac{1}{1-\alpha} - \frac{2}{2-\nu}$. Then

$$\lim_{t\to\infty} \frac{c''(t)}{c'(t)^{1+\frac{2}{1+\gamma}} h(c'(t))} = 0 \quad ,$$

since

$$\frac{t^{\alpha-2}}{t^{(\alpha-1)(1+\frac{2}{1+\gamma})} t^{\delta(\alpha-1)}} = t^{\alpha-2+(1-\alpha)(1+\frac{2}{2-\nu}+\delta)} \to 0 \quad ,$$

the exponent being less than zero. The continuation region therefore satisfies

$$\mathcal{C} = \{(x,t) : |x| < \beta^*(t)\}$$

with

$$\beta^*(t) = (\frac{2}{\nu} c'(t))^{\frac{1}{\nu-2}}(1+o(1)) = (\frac{2\alpha}{\nu})^{\frac{1}{\nu-2}} t^{\frac{1-\alpha}{2-\nu}}(1+o(1)) \quad . \tag{4.26}$$

2.2 $c(t) = t + \log(1+t)$:
The upper bound B for the continuation region fulfills

$$B = \lim_{t\to\infty} \psi(c'(t)) = (\frac{2}{\nu})^{\frac{1}{\nu-2}} \quad .$$

The inner aproximation is given by

$$\beta_-(t) = \psi(c'(t)) = (\frac{2}{\nu}(1 + \frac{1}{1+t}))^{\frac{1}{\nu-2}} \quad .$$

As before we set $h(x) = (x-1)^\delta$ for all $x > 1$ with $0 < \delta < 1$. Then the outer approximation satisfies

$$\beta_+(t) = \psi(c'(t)(1 - h(c'(t))) \quad .$$

With

$$\eta(t) = 1 - h(c'(t)) \quad , \quad f(t) = c'(t)\eta(t)$$

we can verify the condition (cc5), since

$$\lim_{t\to\infty} \frac{B - \beta_-(t)}{B - \beta_+(t)} = \lim_{t\to\infty} \frac{\psi'(c'(t))c''(t)}{\psi'(f(t))f'(t)} = 1 \quad .$$

The last equation follows from $\lim_{t\to\infty} \frac{c''(t)}{f'(t)} = 1$. (cc6) is valid with the same argument as for the previous discussed reward. Hence we obtain, the continuation region is a set

$$\mathcal{C} = \{(x,t) : |x| < \beta^*(t)\}$$

with

$$B - \beta^*(t) = \Big((\frac{2}{\nu})^{\frac{1}{\nu-2}} - (\frac{2}{\nu}(1 + \frac{1}{1+t}))^{\frac{1}{\nu-2}} \Big)(1 + o(1)) \quad .$$

3. $g(x) = |x|^\nu$ with $1 < \nu < 2$:
   This is an example of a convex reward function, and we have to apply Theorem 4.1.7, Corollary 4.1.10 directly. The function $F(y) = G'(y)/U'(y) = \frac{\nu}{2} y^{\nu-2}$ for all $y \in (0, \infty)$ is strictly decreasing from infinity to zero and has as its inverse

$$\psi(z) = (\frac{2}{\nu} z)^{\frac{1}{\nu-2}} \quad , \quad z \in (0, \infty) \quad .$$

   Furthermore $Ag(x) = \frac{1}{2}\nu(\nu-1)x^{\nu-2}$ is decreasing on $(0, \infty)$. Hence the conditions (R1)-(R3) are valid. For cost functions c we have to verify the conditions (cc1)-(cc6).

   3.1 $c(t) = t^\alpha$ with $\frac{\nu}{2} < \alpha < 1$:
   Then (cc1)-(cc3) are of course true. We choose $h(x) = x^\delta$ with $0 < \delta < \frac{1}{1-\alpha} - \frac{2}{2-\nu}$, and the same arguments as in 2.1 provide the conditions (cc4)-(cc6). Hence the continuation region is determined by

$$\mathcal{C} = \{(x,t) : |x| < \beta^*(t)\}$$

   and

$$\beta^*(t) = (\frac{2\alpha}{\nu})^{\frac{1}{\nu-2}} t^{\frac{1-\alpha}{2-\nu}} (1 + o(1)) \quad . \tag{4.27}$$

   3.2 $c(t) = t + \log(1+t)$:
   Then the same conclusions can be drawn as in the preceding example. We obtain

$$\mathcal{C} = \{(x,t) : |x| < \beta^*(t)\}$$

   with

$$B - \beta^*(t) = \Big((\frac{2}{\nu})^{\frac{1}{\nu-2}} - (\frac{2}{\nu}(1 + \frac{1}{1+t}))^{\frac{1}{\nu-2}} \Big)(1 + o(1)) \quad .$$

4. $g(x) = \log(1 + |x|)$:
   This is an example for a concave reward function where Theorem 4.2.2 is

not applicable, but where we can verify the assumptions (R1)-(R3),(cc1)-(cc6) directly. Due to concavity, (R1)-(R3) are true, and the inverse ψ of G'/U' is given by

$$\psi(z) = -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{2z}} \quad .$$

For a cost function c, the inner approximation is defined by

$$\beta_-(t) = -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{2c'(t)}} = \sqrt{\frac{1}{2c'(t)}}(1 + o(1)).$$

We consider $c(t) = t^\alpha$ with $0 < \alpha < 1$ as cost function. Then (cc1)-(cc3) are valid and we choose $h(x) = x^\delta$ with $0 < \delta < \frac{\alpha}{1-\alpha}$. Obviously, (cc4) is satisfied, and the outer approximation fulfills

$$\beta_+(t) = -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{2c'(t)(1 - h(c'(t)))}}$$

which is asymptotically equivalent to $\beta_-(t)$. Hence it remais to prove (cc6). Due to $u(\beta_+(t)) \asymp \psi(c'(t))^2 \asymp \frac{1}{2c'(t))}$, this holds since

$$\frac{c''(t)}{c'(t)^2 h(c'(t))} = O(t^{\alpha-2-(2+\delta)(\alpha-1)}) \quad ,$$

and the exponent is less than zero as $0 < \delta < \frac{\alpha}{1-\alpha}$. Note that $f(t) \asymp g(t)$ means $\lim_{t\to\infty} \frac{f(t)}{g(t)} = 1$. Hence we obtain for the continuation region

$$\mathcal{C} = \{(x,t) : |x| < \beta^*(t)\}$$

with

$$\begin{aligned} \beta^*(t) &= \beta_-(t)(1 + o(1)) = \sqrt{\frac{1}{2c'(t)}}(1 + o(1)) \\ &= \sqrt{\frac{1}{2\alpha} t^{1-\alpha}}(1 + o(1)) \quad . \end{aligned} \tag{4.28}$$

4.2.3 Process of posterior probabilities

We continue the considerations of the example introduced in 2.2.2. We recall that the process of posterior probabilities is a diffusion with generator

$$A = \frac{1}{2}x^2(1-x)^2\partial_x^2$$

on $E = (0,1)$. A symmetric solution w.r.t. $m = \frac{1}{2}$ of $Au = 1$ is given by $u(x) = 2(2x-1)\log(x/(1-x))$ for all $x \in (0,1)$. Thus

$$U(y) = 4y \log \frac{1/2+y}{1/2-y} \quad , \quad U'(y) = 4\log\frac{1/2+y}{1/2-y} + \frac{4y}{(1/2+y)(1/2-y)}.$$

At first we consider concave reward functions and want to obtain an analogous result to Theorem 3.3.4.

4.2.4 Theorem: *Let $g(x) = G(|x-\frac{1}{2}|)$, $x \in (0,1)$, be a reward function with strictly increasing concave C^2-function G satisfying*

$$\lim_{y \to \frac{1}{2}} G'(y)(\frac{1}{2} - y)^{-\gamma} = q$$

for some $q > 0, \gamma \geq 0$. Let c be a cost function, strictly increasing, twice continuously differentiable and concave. Furthermore we assume, $c'(t)$ tends to zero, and the existence of an increasing function $h \geq 0$ with

$$\lim_{x\to 0} h(x) = 0, \lim_{x \to 0} xh'(x) = 0, \lim_{t\to\infty} \frac{c''(t)\log\frac{1}{2c'(t)}}{c'(t)h(c'(t))} = 0 \quad . \tag{4.29}$$

Then the continuation region is the set enscribed between the curves $\frac{1}{2} \pm \beta^*(t)$, i.e.

$$\mathcal{C} = \{(x,t) : |\frac{1}{2} - x| < \beta^*(t)\} \quad ,$$

and

$$\frac{1}{2} - \beta^*(t) = (\frac{2c'(t)}{q})^{\frac{1}{1+\gamma}}(1+o(1)) \quad . \tag{4.30}$$

Proof: We have to examine the conditions (R1)-(R3) and (cc1)-(cc6). Then we can apply Theorem 4.1.7 and Corollary 4.1.10 to obtain the assertion. From the linear case we know that (R1)-(R3) hold. Furthermore the inverse function ψ of G'/U' fulfills

$$\frac{1}{2} - \psi(z) = (\frac{2z}{q})^{\frac{1}{1+\gamma}}(1+o(1)) \qquad \text{for } z \to 0, \tag{4.31}$$

see (3.31). Since E is bounded, concavity implies (cc1)-(cc3) . An inner and outer approximation is given by

$$\beta_-(t) = \psi(c'(t)) \quad , \quad \beta_+(t) = \psi(c'(t)(1-h(c'(t))).$$

From (4.31) we obtain

$$\frac{1}{2} - \beta_-(t) = (\frac{1}{2} - \beta_+(t))(1 + o(1)) = (\frac{2c'(t)}{q})^{\frac{1}{1+\gamma}}(1 + o(1)).$$

Furthermore due to

$$U(y) = 2\log(\frac{1}{\frac{1}{2} - y})(1 + o(1)) \quad \text{for } y \to \frac{1}{2} \quad ,$$

we have

$$\frac{\frac{1}{1+\gamma}\log(\frac{1}{2c'(t)/q})}{U(\beta_+(t))} = \frac{\log\left(\frac{\frac{1}{2}-\beta_+(t)}{(2c'(t)/q)^{\frac{1}{1+\gamma}}}\right) + \log(\frac{1}{\frac{1}{2}-\beta_+(t)})}{U(\beta_+(t))} \to \frac{1}{2} \quad .$$

Hence condition (cc6) is fulfilled if

$$\lim_{t\to\infty} \frac{c''(t)\log\frac{1}{2c'(t)}}{c'(t)h(c'(t))} = 0 \tag{4.32}$$

and the result is proved. □

We want to apply this to some reward and cost functions.

1. $g(x) = G(|x - \frac{1}{2}|)$ with $G(y) = y$:
   This continues our analysis of the stopping problem introduced in 2.3.2. For cost functions c with $\lim_{t\to\infty} c'(t) = 0$

   $$\frac{1}{2} - \beta_-(t) = \frac{1}{2} - \psi(c'(t)) = 2c'(t)(1 + o(1))$$

   compare to (3.33). We consider two cost functions:

   1.1 $c(t) = t^\alpha$ with $0 < \alpha < 1$:
   Then we choose $h(x) = x^\delta$ with $0 < \delta < \frac{1}{1-\alpha}$. (4.29) is satisfied since

   $$t^{\alpha-2+(1-\alpha)(1+\delta)}\log(t^{1-\alpha}) \to 0 \quad .$$

   Thus Theorem 4.2.4 shows

   $$\mathcal{C} = \{(x,t) : |x - \frac{1}{2}| < \beta^*(t)\}$$

   with

   $$\frac{1}{2} - \beta^*(t) = 2c'(t)(1 + o(1)) = 2\alpha t^{\alpha-1}(1 + o(1)) \quad . \tag{4.33}$$

1.2 $c(t) = \log(1+t)$:
Then we choose $h(x) = x^\delta$ with $0 < \delta < 1$ and have

$$\frac{c''(t)\log(\frac{1}{2c'(t)})}{c'(t)h(c'(t))} = -(\frac{1}{1+t})^{1-\delta}\log(\frac{1}{2}(1+t)) \to 0 \quad .$$

Thus the continuation region has the above form with

$$\frac{1}{2} - \beta^*(t) = \frac{2}{1+t}(1+o(1)) \quad . \tag{4.34}$$

2. $g(x) = G(|x - \frac{1}{2}|)$ with $G(y) = -(\frac{1}{2} - y)^\nu$ for $\nu \geq 1$:
Then the assumptions of the preceding theorem are fulfilled for the cost functions $c(t) = t^\alpha$ with $0 < \alpha < 1$ and $c(t) = \log(1+t)$, which can be verified as before. We obtain for the continuation region

$$\mathcal{C} = \{(x,t) : |x - \frac{1}{2}| < \beta^*(t)\}$$

with

$$\frac{1}{2} - \beta^*(t) = (\frac{2c'(t)}{\nu})^{\frac{1}{\nu}}(1+o(1)) = (\frac{2\alpha}{\nu})^{\frac{1}{\nu}}t^{\frac{\alpha-1}{\nu}}(1+o(1)) \tag{4.35}$$

in the first case, and

$$\frac{1}{2} - \beta^*(t) = (\frac{2}{\nu(1+t)})^{\frac{1}{\nu}}(1+o(1)) \tag{4.36}$$

in the latter.

4.2.5 Portfolio optimization

As introduced in 2.2.3 a consideration of portfolio strategies without tansaction costs is related to a diffusion on $E = (0,1)$ with generator

$$A = \frac{1}{2}x^2(1-x)^2\partial_x^2 + x(1-x)(\frac{1}{2} - x)\partial_x \quad .$$

The even solution w.r.t. $m = \frac{1}{2}$ of $Au = 1$ is given by $u(x) = (\log(x/(1-x))^2$. Thus $U(y) = u(\frac{1}{2} + y)$ fulfills

$$U(y) = (\log\frac{\frac{1}{2}+y}{\frac{1}{2}-y})^2 \quad , \quad U'(y) = 2\frac{\log\frac{1/2+y}{1/2-y}}{(1/2+y)(1/2-y)}$$

for all $y \in (0, \frac{1}{2})$.

We consider in the following some examples of reward functions and have to verify the conditions (R1)-(R3) , (cc1)-(cc6) .

1. $g(x) = |\log(x/(1-x))|$:
Then $G(y) = g(\frac{1}{2}+y)$, $y \in (0, \frac{1}{2})$, satisfies

$$G(y) = \log \frac{1/2+y}{1/2-y} \quad , \quad G'(y) = \frac{1}{(1/2+y)(1/2-y)} \quad .$$

Hence $F = G'/U'$ fulfills

$$F(y) = \frac{1}{2}(\log \frac{1/2+y}{1/2-y})^{-1} \quad , y \in (0, \frac{1}{2}) \quad ,$$

and its inverse ψ is defined by

$$\psi(z) = \frac{1}{2} \frac{\exp(\frac{1}{2z}) - 1}{\exp(\frac{1}{2z}) + 1} \qquad , z \in (0, \infty) \quad . \tag{4.37}$$

Thus (R1) and (R2) are valid. Condition (R3) holds, due to $Ag(x) = 0$ for all $x \in (\frac{1}{2}, 1)$.

For a cost function c, the inner and outer approximation are defined by

$$\beta_-(t) = \psi(c'(t)) \quad , \quad \beta_+(t) = \psi(c'(t)(1 - h(c'(t)))$$

where the function h must be chosen, such that the conditions (cc4)-(cc6) are fulfilled.

To examine the condition (cc5) we note

$$\frac{1}{2} - \psi(z) = \exp(-\frac{1}{2z})(1 + o(1)) \quad .$$

Then, with $\eta(t) = 1 - h(c'(t))$, we get

$$\lim_{t\to\infty} \frac{\frac{1}{2} - \beta_+(t)}{\frac{1}{2} - \beta_-(t)} = \lim_{t\to\infty} \exp(\frac{1}{2}(\frac{1}{c'(t)} - \frac{1}{c'(t)\eta(t)})) \quad .$$

The exponent

$$\frac{1}{c'(t)} - \frac{1}{c'(t)\eta(t)} = -\frac{h(c'(t)}{c'(t)(1 - h(c'(t))}$$

converges to zero, if

$$\lim_{x\to 0} \frac{h(x)}{x} = 0 \quad . \tag{4.38}$$

Hence we have proved that, for h with the above additional property (4.38), $\frac{1}{2} - \beta_+(t)$, $\frac{1}{2} - \beta_-(t)$ are asymptotically equivalent.

Secondly, we want to examine which condition on h leads to (cc6). Due to $U(\psi(z)) = (\frac{1}{2z})^2$, it holds

$$U(\beta_+(t)) \asymp (\frac{1}{2c'(t)})^2 \quad .$$

Hence h must satisfy

$$\lim_{t\to\infty} \frac{c''(t)}{h(c'(t))c'(t)^3} = 0 \quad , \tag{4.39}$$

which is the same condition as in the Brownian motion case, see the first example of 4.2.1. So we may use the cost functions c here as well, if (4.38) is additionally fulfilled.

We consider $c(t) = t^\alpha$ with $\frac{2}{3} < \alpha < 1$:
Then $(2\alpha - 1)/(1 - \alpha) > 1$, and we can choose $h(x) = x^\delta$ with $1 < \delta < (2\alpha - 1)/(1 - \alpha)$. As was seen for Brownian motion, the conditions (cc6),(cc4) hold. Furthermore (cc5) is valid, since the additional property (4.38) is fulfilled. We obtain for the continuation region.

$$\mathcal{C} = \{(x,t) : |x - \frac{1}{2}| < \beta^*(t)\}$$

and

$$\begin{aligned} \frac{1}{2} - \beta^*(t) &= \Big(\frac{1}{2} - \psi(c'(t))\Big)(1 + o(1)) \\ &= \exp(-\frac{1}{2\alpha} t^{1-\alpha})(1 + o(1)) \quad . \end{aligned}$$

2. $g(x) = \begin{cases} \log \frac{1}{1-x} & \text{,if } x > \frac{1}{2} \\ \log \frac{1}{x} & \text{,if } x \leq \frac{1}{2} \end{cases}$:

   Then

$$G(y) = -\log(\frac{1}{2} - y) \quad , \quad G'(y) = \frac{1}{1/2 - y}$$

and $F = G'/U'$ fulfills

$$F(y) = (\frac{1}{2} + y)\frac{1}{2\log(\frac{1/2+y}{1/2-y})} \tag{4.40}$$

which is decreasing on $(0, \frac{1}{2})$ with $\lim_{y\to 0} F(y) = \infty$, $\lim_{y\to 1/2} F(y) = 0$. Hence (R1),(R2) are valid. Due to $Ag(x) = \frac{1}{2}x(1-x)$, the condition (R3) follows.

As in the preceding example, the asymptotic equivalence of the inner and outer approximation has to be examined carefully. We define

$$F_1(y) = -\frac{1}{2\log(\frac{1}{2}-y)} \quad , y \in (0, \frac{1}{2}) \quad , \tag{4.41}$$

and note

$$\lim_{y\to\frac{1}{2}} \frac{F(y)}{F_1(y)} = 1 \quad . \tag{4.42}$$

The inverse function of F_1, denoted by ψ_1, is given by

$$\psi_1(z) = \frac{1}{2} - \exp(-\frac{1}{2z}) \quad , \quad \psi_1'(z) = -\frac{1}{2}\frac{1}{z^2}\exp(-\frac{1}{2z}), \tag{4.43}$$

$z \in (0, \infty)$. It is easy to see

$$\lim_{y\to\frac{1}{2}} \psi_1'(F(y))F'(y) = 1 \quad ,$$

which implies

$$\lim_{z\to 0} \frac{1/2 - \psi_1(z)}{1/2 - \psi(z)} = 1 \quad , \tag{4.44}$$

since

$$\lim_{z\to 0} \frac{1/2 - \psi_1(z)}{1/2 - \psi(z)} = \lim_{z\to 0} \frac{\psi_1'(z)}{\psi'(z)} = \lim_{y\to\frac{1}{2}} \frac{\psi_1'(F(y))}{\psi'(F(y))} = \lim_{y\to\frac{1}{2}} \psi_1'(F(y))F'(y) \quad .$$

With $\eta(t) = 1 - h(c'(t))$ we obtain

$$\begin{aligned} \lim_{t\to\infty} \frac{\frac{1}{2} - \beta_-(t)}{\frac{1}{2} - \beta_+(t)} &= \lim_{t\to\infty} \frac{1/2 - \psi(c'(t))}{1/2 - \psi(c'(t)\eta(t))} \\ &= \lim_{t\to\infty} \frac{1/2 - \psi_1(c'(t))}{1/2 - \psi_1(c'(t)\eta(t))} \\ &= \lim_{t\to\infty} \exp(-\frac{1}{2}(\frac{1}{c'(t)} - \frac{1}{c'(t)\eta(t)})) = 1 \quad , \end{aligned}$$

if $\lim_{x\to 0} \frac{h(x)}{x} = 0$.

Secondly we have to investigate condition (cc6). Due to (4.41) and (4.42)

$$c'(t) = F(\beta_-(t)) \asymp -\frac{1}{2\log(\frac{1}{2} - \beta_-(t))} \quad ,$$

and

$$c'(t)(1 - h(c'(t))) = F(\beta_+(t)) \asymp -\frac{1}{2\log(\frac{1}{2} - \beta_+(t))} \quad .$$

Hence

$$\log(\frac{1}{2} - \beta_-(t)) \asymp \log(\frac{1}{2} - \beta_+(t)) \quad , \tag{4.45}$$

and from $U(y) \asymp \left(\log(\frac{1}{2} - y)\right)^2$ for $y \to \frac{1}{2}$ we obtain

$$U(\beta_+(t)) \asymp U(\beta_-(t)) \quad .$$

Furthermore, (4.44) shows

$$U(\beta_-(t)) = \left(\log(\frac{1}{2} - \psi_1(c'(t)))\right)^2 + o(1) = (\frac{1}{2c'(t)})^2 + o(1) \quad .$$

Thus condition (cc6) holds, if h fulfills

$$\lim_{t\to\infty} \frac{c''(t)}{c'(t)^3 h(c'(t))} = 0 \quad . \tag{4.46}$$

Hence, for $c(t) = t^\alpha$ with $\frac{2}{3} < \alpha < 1$ we can choose $h(x) = x^\delta$ with $1 < \delta < \frac{2\alpha-1}{1-\alpha}$. Then the continuation region is given as

$$\mathcal{C} = \{(x,t) : |x - \frac{1}{2}| < \beta^*(t)\} \quad ,$$

with

$$\begin{aligned} \frac{1}{2} - \beta^*(t) &= (\frac{1}{2} - \psi(c'(t)))(1 + o(1)) \\ &= (\frac{1}{2} - \psi_1(c'(t)))(1 + o(1)) \\ &= \exp(-\frac{1}{2c'(t)})(1 + o(1)) \\ &= \exp(-\frac{t^{1-\alpha}}{2\alpha})(1 + o(1)) \quad . \end{aligned}$$

Note that the boundaries of the continuation region in both examples have the same asymptotical shape.

Chapter 5

Convex costs

5.1 Asymptotics of the continuation region

As before we consider a one-dimensional symmetric A-diffusion on an open interval $E = (m-l, m+l)$. For a symmetric reward function g and a convex cost function c, the optimal stopping problem corresponding to the payoff

$$(x,t) \to g(x) - c(t)$$

will be treated in this chapter. The case of convex growth of costs for observations leads to a different behaviour of the continuation region than in the concave case. Since the cost rate increases, one expects that the continuation region shrinks and is contained in the set

$$\mathcal{C}_{out} = \{(x,t) : |x-m| < \psi(c'(t))\} \quad .$$

Thus, different to the concave case, an outer approximation should be easily obtainable by applying the results of the linear case to the increasing cost rate function.

It will be more difficult to construct an inner approximation. The main idea is to find an appropriate subharmonic function. This will be explained in the following. But let us first state the conditions which the cost function $c : [0,\infty) \to [0,\infty)$ should fulfill.

(cv1) c is strictly increasing with $c(0) \geq 0$.

(cv2) c is twice continuously differentiable and convex .

(cv3) For all $k > 0$ and all $x \in E$

$$E_x \sup_{t \geq 0}(g(X_t) - kt) < \infty.$$

Due to convexity, (cv3) implies

$$E_x \sup_{t \geq 0}(g(X_t) - c(t_0 + t)) < \infty$$

for all $x \in E$, $t_0 \geq 0$. Furthermore, as in the linear case

$$\lim_{t \to \infty} g(X_t) - c(t_0 + t) = -\infty \quad P_x - \text{ a.s.}$$

for all $x \in E$. Thus we can apply the usual theory of optimal stopping and obtain that the continuation region $\mathcal{C}$ is an open set, and that the first exit time from $\mathcal{C}$ is an optimal stopping time, see Theorem 4.1.1. If additionally the coefficients of A are locally Hölder continuous, the optimal value function v fulfills the partial differential equation

$$(\partial_t + A)v = 0 \quad \text{on } \mathcal{C} \quad .$$

Introducing the difference $d(x,t) = v(x,t) - (g(x) - c(t))$, we get the analogous result as in the concave case.

5.1.1 Lemma: *If (cv1)-(cv3) are fulfilled with a symmetric continuous reward function g, then*

(i) $v(x, \cdot)$ *is decreasing for all* $x \in E$,

(ii) $d(x, \cdot)$ *is decreasing for all* $x \in E$,

(iii) $v(\cdot, t)$ *is even w.r.t.* m *for all* $t \geq 0$, *and the continuation region is symmetric w.r.t.* m, *i.e.*

$$(m + y, t) \in \mathcal{C} \Longleftrightarrow (m - y, t) \in \mathcal{C} \quad \textit{for all } y \in (0, l), t \geq 0 \quad .$$

Proof: We may argue as in the proof of Lemma 4.1.2 by using convexity instead of concavity. □

For the further analysis of the stopping problem we assume the conditions (R1)-(R3) to hold for the symmetric continuous reward function g. Then, as in the preceding chapter, we can apply the results for linear cost functions for all cost rates $k > 0$. Here, in the case of convex costs, this leads to an outer approximation of the continuation region. We introduce the function ζ_+ by

$$\zeta_+(t) = \psi(c'(t)) \quad \text{for all } t \geq 0 \tag{5.1}$$

and recall that $m \pm \zeta_+(t)$ are the unique solutions of

$$g'(x) = c'(t)u(x) \tag{5.2}$$

in E, compare to Chapter 3. The function ψ, implicitly defined by (5.2), is the inverse of the strictly decreasing function G'/U' and therefore strictly decreasing itsself. Together with the convexity of c, this shows that ζ_+ is a strictly decreasing function. Furthermore we can state

5.1.2 Lemma: *If g fulfills (R1)-(R3) and c satisfies (cv1)-(cv3), then the continuation region $\mathcal{C}$ is contained in*

$$\mathcal{C}_{out} = \{(x,t) : |x - m| < \zeta_+(t)\} \quad .$$

Proof: We follow the concave case and replace the convex cost function by its linear tangent at each $t_0 > 0$. Convexity implies

$$c(t_0 + t) \geq c'(t_0)t + c(t_0) \quad \text{for all } t > 0.$$

If a point $(x, t_0) \in \mathcal{C}$ is contained in the continuation region we find a stopping time τ such that

$$\begin{aligned} g(x) - c(t_0) &< E_x\left(g(X_\tau) - c(t_0 + \tau)\right) \\ &\leq E_x(g(X_\tau) - c'(t_0)\tau) - c(t_0) \quad . \end{aligned}$$

Thus x lies in the continuation region for linear costs with cost rate $c'(t_0)$, and we obtain

$$|x - m| < \psi(c'(t_0)) = \zeta_+(t_0) \quad .$$

Hence the assertion is valid. □

To establish that the continuation region is a set enscribed between boundary curves $(m \pm \beta^*(t))_{t \geq 0}$ becomes more difficult in the convex case . Helpful is a first inner approximation which can easily be obtained. For this purpose we consider the equation

$$Ag(x) = c'(t) \quad . \tag{5.3}$$

If, for $t > 0$, the above equation is solvable in x, (R3) implies that it has exactly two symmetric solutions, denoted by $m \pm \delta(t)$. If no solution exists we put $\delta(t) = 0$. Since Ag is decreasing on $(m, m+l)$ and c' is increasing in t we have thus defined a continuous decreasing curve δ, such that

$$Ag(x) > c'(t) \quad \text{for all } 0 < |x - m| < \delta(t) \quad . \tag{5.4}$$

This states that $(x,t) \to g(x) - c(t)$ is subharmonic in the region $\{(x,t) : m < x < m + \delta(t)\} \cup \{(x,t) : m - \delta(t) < x < m\}$, and it is not surprising that the assertion of the following lemma holds

5.1.3 Lemma: *Let c satisfy (cv1)-(cv3), and let g fulfill (R1)-(R3) . Then*

$$\{(x,t) : 0 < |m - x| < \delta(t)\}$$

is contained in the continuation region.

Proof: This follows easily from the above inequality (5.4). For a point (x_0, t_0) with $m < x_0 < m + \delta(t_0)$ we may choose a neighbourhood $\Gamma = \{(x,t) : |x - x_0| < \epsilon, |t - t_0| < \epsilon\}$ such that

$$(\partial_t + A)\hat{g}(x,t) > 0 \quad \text{on} \ \ \Gamma$$

with $\hat{g}$ denoting the payoff function $\hat{g}(x,t) = g(x) - c(t)$. On Γ, $\hat{g}$ is subharmonic, and therefore the first exit time from Γ should bring an improvement compared to immediate stopping. We put

$$\tau = \inf\{t \geq 0 : (X_t, t_0 + t) \notin \Gamma\} \quad .$$

Then

$$\begin{aligned} E_{x_0}\left(g(X_\tau) - c(t_0 + \tau)\right) &= E_{x_0}\, \hat{g}(X_\tau, t_0 + \tau) \\ &= \hat{g}(x_0, t_0) + E_{x_0} \int_0^\tau (\partial_t + A)\hat{g}(X_t, t_0 + t)dt \\ &> \hat{g}(x_0, t_0) = g(x_0) - c(t_0) \end{aligned}$$

and therefore (x_0, t_0) is contained in the continuation region. □

Note, that we give no statement about the midpoint line $\{(m,t) : t \geq 0\}$ since g might not be differentiable in m. Furthermore, the above inner approximation is not very accurate. In the case $Ag \leq 0$ it even collapses to the empty set since $\delta(t) = 0$ for all t. But still it provides additional information that can be used to derive the shape of the continuation region. As will be explained later different arguments are necessary to decide whether the midpoint line belongs to the continuation region.

5.1.4 Theorem: *Let g fulfill (R1)-(R3), and let c satisfy (cv1)-(cv3). Then there exists a decreasing function $\beta^* : [0,\infty) \to (0,\infty)$ such that the continuation region fulfills*

$$\mathcal{C} \setminus \{(m,t) : t \geq 0\} = \{(x,t) : 0 < |x-m| < \beta^*(t)\} \quad .$$

Proof: We define

$$\beta^*(t) = \inf\{y > 0 : (m+y,t) \notin \mathcal{C}\} = \inf\{y > 0 : d(m+y,t) = 0\} \tag{5.5}$$

with the convention $\inf \emptyset = 0$. Then β^* is decreasing since $d(x,\cdot)$ is decreasing for all $x \in E$, and by definition $\{(x,t) : 0 < |x-m| < \beta^*(t)\} \subset \mathcal{C}$. From Lemma 5.1.3, $\delta(t) \leq \beta^*(t)$ for all $t > 0$. It remains to prove

$$\{(x,t) : |x-m| > \beta^*(t)\} \subset \mathcal{E}$$

with $\mathcal{E}$ denoting the stopping region. If $|x-m| \geq \zeta_+(t)$, this is true due to the outer approximation, and it remains to consider x such that $|m-\beta^*(t)| < x < |m-\zeta_+(t)|$. We first assume $m+\beta^*(t) < x < m+\zeta_+(t)$ and have to show that immediate stopping is optimal. The first exit time τ^* from $\mathcal{C}$ is optimal, hence

$$v(x,t) = E_x(g(X_{\tau^*}) - c(t+\tau^*)) \quad .$$

We introduce the first time σ that the diffusion hits one of the moving boundaries $(m+\beta^*(t+s))_{s\geq 0}$, $(m+\zeta_+(t+s))_{s\geq 0}$ by

$$\sigma = \inf\{s \geq 0 : X_s = m+\beta^*(t+s) \text{ or } \; X_s = m+\zeta_+(t+s)\}$$

Then, since $m+\beta^*(t) < x < m+\zeta_+(t)$, the diffusion will first exit from $\mathcal{C}$ before it can reach one of the moving boundaries, i.e. $\tau^* \leq \sigma$. Contrary to the previous chapters we will use a localization argument at m and not at the boundary of E. This is necessary since g might not be differentiable at m. Therefore we introduce the sequence of stopping times σ_n defined by

$$\sigma_n = \inf\{s \geq 0 : X_s \leq m + \frac{1}{n}\} \tag{5.6}$$

Then dominated convergence implies

$$E_x(g(X_{\tau^*}) - c(t+\tau^*)) = \lim_{n\to\infty} E_x\left(g(X_{\tau^*\wedge\sigma_n}) - c(t+\tau^*\wedge\sigma_n)\right) \quad . \tag{5.7}$$

Note that $\delta(t+s) \leq \zeta_+(t+s)$ implying $Ag(X_s) \leq c'(t+s)$ for all $s \leq \tau^* \wedge \sigma_n$. Due to $X_s \in [m+1/n, m+\zeta_+(t)]$ for $s \leq \tau^* \wedge \sigma_n$ this yields with Prop. 2.1.2

$$\begin{aligned} E_x g(X_{\tau^*\wedge\sigma_n}) &= g(x) + E_x \int_0^{\tau^*\wedge\sigma_n} Ag(X_s)ds \\ &\leq g(x) + E_x \int_0^{\tau^*\wedge\sigma_n} c'(t+s)ds \\ &= g(x) + E_x\, c(t+\tau^*\wedge\sigma_n) - c(t) \end{aligned} \tag{5.8}$$

Thus, by inserting into (5.7)

$$v(x,t) = E_x\, g(X_{\tau^*}) - c(t+\tau^*) \leq g(x) - c(t) \quad .$$

The case $m - \zeta_+(t) < x < m - \beta^*(t)$ follows by symmetry and the assertion is proved. □

Note that the arguments in the preceding proof do not work when starting from the midpoint m. The reason is that g need not be differentiable in m. It is only a C^2-function on $(m, m+l)$ respectively $(m-l, m)$. Thus the procedure in (5.8) is only correct if the diffusion remains in one of the sets $(m-l, m)$ or $(m, m+l)$ until stopping. Later we will see with other arguments that the midpoint line belongs to the continuation region.

To improve the inner approximation requires the following additional assumptions. Recall that $\psi : (0,\infty) \to (0,l)$ denotes the inverse function of G'/U' and that both functions are decreasing. Furthermore, as in the concave case, we define $c'(\infty) = \lim_{t\to\infty} c'(t)$, $B = \lim_{t\to\infty} \psi(c'(t))$ It will turn out that the real number $B \in [0,\infty)$ provides a lower bound for the continuation region, i.e. $(m-B, m+B) \times [0,\infty) \subset \mathcal{C}$.

We demand the following properties for a decreasing differentiable function $h \geq 0$.

(cv4) $\lim_{x\uparrow c'(\infty)} h(x) = 0 \quad , \quad \lim_{x\uparrow c'(\infty)} xh'(x) = 0 \quad ,$

(cv5) the curve ζ_- defined by $\zeta_-(t) = \psi(c'(t)(1 + h(c'(t))))$ for all $t \geq 0$ is asymptotically equivalent to ζ_+ , i.e.

$$\lim_{t\to\infty} \frac{\zeta_-(t) - B}{\zeta_+(t) - B} = 1 \quad ,$$

(cv6)

$$\lim_{t\to\infty} \frac{c''(t)U(\zeta_+(t))}{h(c'(t))c'(t)} = 0 \quad .$$

Since ψ decreases, $\zeta_-(t) \leq \zeta_+(t)$ for all $t \geq 0$. The aim is to show $\zeta_-(t) \leq \beta^*(t) \leq \zeta_+(t)$ for large t. In a first step we will contruct a majorant of the payoff function, subharmonic for large t.

5.1.5 Lemma: *Let the reward g fulfill (R1)-(R3), and let the cost function c satisfy (cv1)-(cv6). Then there exists a function*

$$\phi : E \times (0, \infty) \to I\!R$$

and some $t_0 > 0$ such that the following properties hold:

(i) $(\partial_t + A)\phi(x, t) \geq 0$ *for all $x \in E, t \geq t_0$.*

(ii) ϕ is even w.r.t. the midpoint m.

(iii) $\phi(x, t) \geq g(x) - c(t)$ for all $x \in E$, $t > 0$.

(iv) $\phi(m \pm \zeta_-(t), t) = g(m \pm \zeta_-(t)) - c(t)$ for all $t > 0$.

(v) ϕ is bounded from above on $\{(x, t) : t \geq t_0, |m - x| \leq \zeta_-(t)\}$.

Proof: Recall that u denotes the symmetric solution of $Au = 1$, vanishing at m, and that u is non-negative. Furthermore we define

$$\eta(t) = 1 + h(c'(t)) \quad , \quad f(t) = \eta(t)c'(t) \quad , t > 0, \tag{5.9}$$

and

$$\phi_1(x, t) = \eta(t)c'(t)u(x) - c(t) \quad , t > 0, x \in E \quad . \tag{5.10}$$

Then

$$f'(t) = \eta'(t)c'(t) + \eta(t)c''(t) = c''(t)\Big(1 + h(c'(t)) + c'(t)h'(c'(t))\Big)$$

together with (cv4) shows the existence of t_0 such that $f'(t) \geq 0$ for $t \geq t_0$. This implies that ϕ_1 is subharmonic for $t \geq t_0$, since

$$\begin{aligned}(\partial_t + A)\phi_1(x,t) &= f'(t)u(x) + c'(t)(\eta(t) - 1) \\ &= f'(t)u(x) + c'(t)h(c'(t)) \quad .\end{aligned} \tag{5.11}$$

We lift ϕ_1 in such a way that it exceeds $(x,t) \to g(x) - c(t)$ and touches it at $m \pm \zeta_-(t)$. Therefore we define

$$\phi(x,t) = \phi_1(x,t) + \gamma(t)$$

with $\gamma(t) = g(m + \zeta_-(t)) - c'(t)\eta(t)u(m + \zeta_-(t))$. Then ϕ fulfills the properties (ii)-(iv) since $x \to g(x) - c'(t)\eta(t)u(x)$ has its maximal points at $m \pm \zeta_-(t)$.

To prove (i) we verify, as in the concave case, that

$$\gamma'(t) = -f'(t)u(m + \zeta_-(t))$$

and

$$(\partial_t + A)\phi(x,t) = f'(t)u(x) + c'(t)h(c'(t)) - f'(t)u(m + \zeta_-(t)) \quad .$$

Since $f'(t) \geq 0$ for large t we have to verify that $f'(t)u(m + \zeta_-(t))$ tends faster to zero than $c'(t)h(c'(t))$. But this follows from $f'(t) = c''(t)(1 + o(1))$ and condition (cv6).

To show the boundedness condition (v) we note that

$$\begin{aligned}\partial_t\phi(x,t) &= f'(t)u(x) - c'(t) + \gamma'(t) \\ &= f'(t)(u(x) - u(m + \zeta_-(t))) - c'(t) \quad .\end{aligned} \tag{5.12}$$

Thus, if $|m - x| < \zeta_-(t)$, then

$$\partial_s\phi(x,s) \leq 0 \quad \text{for all } s \in (t_0, t)$$

and

$$\phi(x,t) \leq \phi(x,t_0) \leq \phi(m + \zeta_-(t_0), t_0)$$

for all $t > t_0$, $|m - x| < \zeta_-(t)$. $\square$

With the help of this lemma a suitable inner approximation of the continuation region can be derived. We recall that $\zeta_-(t)$ is the unique solution of

$$G'(y) = c'(t)(1 + h(c'(t))U'(y)$$

for y on $(0, l)$.

5.1.6 Theorem: *Let the reward g fulfill (R1)-(R3), and let the cost function c satisfy (cv1)-(cv6).Then there exists some $t_0 > 0$ such that*

$$\mathcal{C}_{in} = \{(x,t) : 0 \leq |x - m| < \zeta_-(t), t \geq t_0\}$$

is contained in the continuation region $\mathcal{C}$.

Proof: From Lemma 5.1.5 there exists a function $\phi : E \times (0,\infty) \to (0,\infty)$ and some $t_0 > 0$ such that the properties (i)-(v) are fulfilled.

For $t \geq t_0$ and $|x - m| < \zeta_-(t)$ we want to verify that (x,t) lies in the continuation region. We define the first hitting time of the moving boundaries $m \pm \zeta_-(t + \cdot)$ by

$$\begin{aligned} \tau &= \inf\{s \geq 0 : |X_s - m| = \zeta_-(t+s)\} \\ &= \inf\{s \geq 0 : \phi(X_s, t+s) = g(X_s) - c(t+s)\} \quad . \end{aligned} \tag{5.13}$$

Then

$$v(x,t) \geq E_x\left(g(X_\tau) - c(t+\tau)\right) = E_x\, \phi(X_\tau, t+\tau) \quad . \tag{5.14}$$

Furthermore, for each fixed $T > 0$, $\phi(X_{\tau\wedge T}, t + \tau \wedge T)_{T\geq 0}$ is bounded from above due to property (v). Hence Fatou's lemma yields

$$E_x\, \phi(X_\tau, t+\tau) \geq \limsup_{T\to\infty} E_x\, \phi(X_{\tau\wedge T}, t + \tau\wedge T) \quad , \tag{5.15}$$

and optional sampling provides, due to $(X_s, t+s) \in (m - \zeta_-(t), m + \zeta_-(t)) \times (t, t+T)$ for all $s \leq T \wedge \tau$,

$$\begin{aligned} E_x\, \phi(X_{\tau\wedge T}, t+\tau\wedge T) &= \phi(x,t) + E_x \int_0^{\tau\wedge T} (\partial_t + A)\phi(X_s, t+s)ds \\ &\geq \phi(x,t) > g(x) - c(t) \quad . \end{aligned}$$

Together with (5.14) this shows that (x,t) is contained in the continuation region, and the inner approximation is obtained. □

In extension of Theorem 5.1.4, our additional assumptions (cv4)-(cv6) provide that $\{(m,t) : t \geq t_0\}$ is contained in the continuation region. Since the difference $v(x,t) - (g(x) - c(t))$ is decreasing in time, the whole line $\{(m,t) : t \geq 0\}$ belongs to $\mathcal{C}$. In view of Theorem 5.1.4 we get an improved version.

5.1.7 Corollary: *Under the assumptions of the preceding theorem there exists a decreasing function $\beta^* : [0,\infty) \to (0,l]$ such that the continuation region $\mathcal{C}$ fulfills*

$$\mathcal{C} = \{(x,t) : 0 \leq |x-m| < \beta^*(t)\} \quad .$$

Furthermore β^* is continuous from the right and fulfills

$$0 < \beta^*(t) \leq \zeta_+(t)$$

for all $t \geq 0$.

Proof: It remains to show continuity from the right. For this let (t_n) be a sequence decreasing to t. Then $\beta^*(t_n)$ is increasing and tends to $\beta^*(t+)$ where $\beta^*(t+) \leq \beta^*(t)$. Since the stopping region $\mathcal{E}$ is closed, $(m + \beta^*(t+), t)$ is contained in $\mathcal{E}$ as limit of $(\beta^*(t_n), t_n)$ in $\mathcal{E}$. Hence $\beta^*(t) \leq \beta^*(t+)$ giving equality. The function β^* is strictly positive since it exceeds ζ_- for large t due to the inner approximation. □

We have thus shown that, as in the concave case, the continuation set is the region between two curves $m \pm \beta^*(t)$, and we have determined its asymptotic shape. Recall $B = \lim_{t\to\infty} \psi(c'(t))$.

5.1.8 Corollary: *If c fulfills (cv1)-(cv6) and g satisfies (R1)-(R3), then the boundary of the continuation region is asymtotically equivalent to its inner and outer approximation, i.e.*

$$\lim_{t\to\infty} \frac{\zeta_+(t) - B}{\beta^*(t) - B} = \lim_{t\to\infty} \frac{\zeta_-(t) - B}{\beta^*(t) - B} = 1 \quad .$$

Proof: The proof is immediate due to the inner and outer approximation and the fact that ζ_- and ζ_+ are asymptotically equivalent. □

5.2 Applications

We investigate our three examples from the previous chapters.

5.2.1 Brownian motion

The differential generator is $A = \frac{1}{2}\partial_x^2$, and $u(x) = x^2$ is the even solution of $Au = 1$, vanishing at zero.

At first we treat concave reward functions. Our aim is to derive an analogous result to Theorem 4.2.2

5.2.2 Theorem: *Let $g(x) = G(|x|)$ be a reward with strictly increasing concave C^2-function G satisfying*

$$\lim_{y\to 0} G'(y)y^\gamma = q \qquad \textit{for some} \quad q > 0, \gamma \geq 0.$$

Let c be a cost function with $\lim_{t\to\infty} c'(t) = \infty$ that is strictly increasing, twice continuously differentiable and convex. Furthermore we assume the existence of a decreasing function $h \geq 0$ such that

$$\lim_{x\to\infty} h(x) = 0, \lim_{x\to\infty} xh'(x) = 0, \lim_{t\to\infty} \frac{c''(t)c'(t)^{-\frac{2}{1+\gamma}}}{c'(t)h(c'(t))} = 0. \tag{5.16}$$

Then the continuation region is a set enscribed between two curves $\pm\beta^*(t)$, i.e.

$$\mathcal{C} = \{(x,t) : |x| < \beta^*(t)\}$$

with

$$\beta^*(t) = \psi(c'(t))(1 + o(1)) = (\frac{2}{q}c'(t))^{-\frac{1}{1+\gamma}}(1 + o(1)) \quad . \tag{5.17}$$

Proof: We have to examine the conditions (R1)-(R3) and (cv1)-(cv6). Then Corollary 5.1.7 , 5.1.8 will provide the assertion.
Since G is concave and U is convex, the conditions (R1)-(R3) hold as was seen in Proposition 4.1.3. Furthermore, condition (cv3) is valid, and the results for linear cost functions are applicable. The function $F = G'/U'$ has a decreasing inverse function ψ satisfying $\lim_{z\to 0}\psi(z) = +\infty$, $\lim_{z\to\infty}\psi(z) = 0$. The inner and outer approximations

$$\zeta_+(t) = \psi(c'(t)) \quad , \quad \zeta_-(t) = \psi(c'(t)(1 + h(c'(t)))$$

fulfill, due to (3.25),

$$\begin{aligned} \zeta_+(t) &= (\frac{2}{q}c'(t))^{-\frac{1}{1+\gamma}}(1+o(1)) \\ \zeta_-(t) &= (\frac{2}{q}c'(t)(1+h(c'(t))))^{-\frac{1}{1+\gamma}}(1+o(1)) \quad . \end{aligned} \tag{5.18}$$

Thus they are asymptotically equivalent, and (cv5) is fulfilled. Finally we note that (5.16) implies (cv6), since $u(\zeta_+(t)) = O(c'(t)^{-\frac{2}{1+\gamma}})$. □

We want to apply this to some reward and cost functions to see how the continuation region shrinks.

1. $g(x) = |x|$:
   Then $\psi(z) = \frac{1}{2z}$ for $z > 0$, and the outer approximation satisfies

   $$\zeta_+(t) = \frac{1}{2c'(t)} \qquad \text{for all } t > 0 \quad .$$

   We now consider several different cost functions:

   1.1 $c(t) = t^\alpha$ with $\alpha > 1$:
   We have to determine a function h that satisfies (5.16). For this let $h(x) = x^{-\delta}$ with $0 < \delta < \frac{2\alpha-1}{\alpha-1}$. Then the first and second equation of (5.16) hold. The third follows from $\gamma = 0$ and

   $$\frac{t^{\alpha-2}t^{-2(\alpha-1)}}{t^{\alpha-1}t^{-\delta(\alpha-1)}} = t^{\alpha-2-(\alpha-1)(3-\delta)} \to 0 \quad ,$$

   since the exponent is less than zero . Hence the continuation region satisfies

   $$\mathcal{C} = \{(x,t) : |x| < \beta^*(t)\}$$

   with

   $$\beta^*(t) = \frac{1}{2c'(t)}(1+o(1)) = \frac{1}{2\alpha}t^{-(\alpha-1)}(1+o(1)) \quad . \tag{5.19}$$

   1.2 $c(t) = t^\alpha \log(1+t)$:
   Then we choose $h(x) = x^{-\delta}$ with $0 < \delta \leq \frac{2\alpha-1}{\alpha-1}$ and, as in the previous example, we obtain a continuation region with boundary fulfilling

   $$\beta^*(t) = \frac{1}{2\alpha}\frac{t^{1-\alpha}}{\log(1+t)}(1+o(1)) \quad .$$

1.3 $c(t) = e^{\lambda t}$ with $\lambda > 0$:
Again we use $h(x) = x^{-\delta}$ with $0 < \delta < 2$. Then

$$\frac{c''(t)}{c'(t)^3 h(c'(t))} = \lambda^{\delta-1} \exp(t\lambda(1+\delta-3)) \to 0 \quad .$$

Hence Theorem 5.2.2 provides

$$\mathcal{C} = \{(x,t) : |x| < \beta^*(t)\}$$

with

$$\beta^*(t) = \frac{1}{2\lambda} e^{-\lambda t}(1+o(1)) \quad . \tag{5.20}$$

1.4 $c(t) = t - \log(1+t)$:
Then $c'(t) = 1 - \frac{1}{1+t}$ increases to one and $c''(t) = \frac{1}{(1+t)^2}$. Theorem 5.2.2 is not directly applicable since the cost rate does not tend to infinity. But the inner and outer approximation can be determined explicitly, and we can verify whether Corollary 5.1.7 , 5.1.8 are applicable. The outer approximation fulfills

$$\zeta_+(t) = \frac{1}{2c'(t)} = \frac{1}{2} + \frac{1}{2t} \quad .$$

We let $h(x) = (1-x)^\delta$ for all $0 < x < 1$ with $1 < \delta < 2$ and obtain the inner approximation

$$\zeta_-(t) = \frac{1}{2c'(t)(1+h(c'(t)))} = \frac{1}{2} + \frac{(1+t)^\delta - t}{2t((1+t)^\delta + 1))} \quad . \tag{5.21}$$

$\delta > 1$ implies

$$\lim_{t\to\infty} \frac{\zeta_+(t) - \frac{1}{2}}{\zeta_-(t) - \frac{1}{2}} = 1 \quad . \tag{5.22}$$

It remains to examine condition (cv6). But this can be done as in the concave case. Hence we obtain the continuation region

$$\mathcal{C} = \{(x,t) : |x| < \beta^*(t)\}$$

with

$$\beta^*(t) = \frac{1}{2} + \frac{1}{2t}(1+o(1)) \quad . \tag{5.23}$$

2. $g(x) = x^\nu$ with $0 < \nu < 2$:
Then $F(y) = G'(y)/U'(y) = \frac{\nu}{2} y^{\nu-2}$ has the decreasing inverse

$$\psi(z) = (\frac{2}{\nu} z)^{\frac{1}{\nu-2}} \qquad , z \in (0, \infty).$$

For a convex cost function c, the inner and outer approximation are given by

$$\begin{aligned} \zeta_+(t) = \psi(c'(t)) = (\frac{2}{\nu} c'(t))^{\frac{1}{\nu-2}} \quad , \\ \zeta_-(t) = (\frac{2}{\nu} c'(t)(1 + h(c'(t))))^{\frac{1}{\nu-2}} \quad . \end{aligned}$$

We consider

2.1 $c(t) = t^\alpha$ with $\alpha > 1$:
Then we choose $h(x) = x^{-\delta}$ with $0 < \delta < \frac{1}{\alpha-1} + \frac{2}{2-\nu}$ and note that

$$\frac{c''(t) U(\zeta_+(t))}{c'(t) h(c'(t))} = O(t^{\alpha-2+(\alpha-1)(\frac{2}{\nu-2}-1)+\delta}) \quad .$$

Since the exponent is less than zero condition (cv6) is fulfilled. We obtain that the continuation region shrinks as

$$\beta^*(t) = (\frac{2}{\nu} c'(t))^{\frac{1}{\nu-2}} (1 + o(1)) = (\frac{2\alpha}{\nu})^{\frac{1}{\nu-2}} t^{\frac{\alpha-1}{\nu-2}} (1 + o(1)) \quad . \tag{5.24}$$

2.2 $c(t) = e^{-\lambda t}$ with $\lambda > 0$:
Again we use $h(x) = x^{-\delta}$ with $0 < \delta < \frac{2}{2-\nu}$ where

$$\frac{c''(t) U(\zeta_+(t))}{c'(t) h(c'(t))} = O(\exp(t(\frac{2\lambda}{\nu - 2} + \lambda\delta))$$

and the exponent is less than zero. We obtain that the continuation region can be enscribed between the curves $\pm\beta^*(t)$ and

$$\begin{aligned} \beta^*(t) &= (\frac{2}{\nu} c'(t))^{\frac{1}{\nu-2}} (1 + o(1)) \\ &= (\frac{2}{\nu})^{\frac{1}{\nu-2}} \exp(-\frac{\lambda}{2-\nu} t)(1 + o(1)) \quad . \end{aligned} \tag{5.25}$$

5.2.3 Process of posterior probabilities

Continuing the example introduced in Chapter 2.2.2, we consider a symmetric diffusion on $E = (0, 1)$ with generator

$$A = \frac{1}{2}x^2(1-x)^2\partial_x^2 \quad .$$

We recall that the symmetric solution of $Au = 1$ vanishing at $m = \frac{1}{2}$ is given by $u(x) = 2(2x-1)\log(x/(1-x))$ and $U(y) = u(\frac{1}{2}+y)$ satisfies

$$U(y) = 4y\log\frac{1/2+y}{1/2-y} \quad , \quad U'(y) = 4\log\frac{1/2+y}{1/2-y} + \frac{4y}{(1/2+y)(1/2-y)} \quad .$$

For concave reward functions we can state an analogous result to Theorem 4.2.4.

5.2.4 Theorem: *Let $g(x) = G(|x-1/2|)$ for all $x \in (0,1)$ be a reward with strictly increasing concave C^2-function G satisfying*

$$\lim_{y\to 0} G'(y)y^\gamma = q \qquad \text{for some } q > 0, \gamma \geq 0 \quad . \tag{5.26}$$

Let c be a cost function with $\lim_{t\to\infty} c'(t) = \infty$, strictly increasing, twice continuously differentiable and convex. Furthermore we assume that there exists a decreasing function $h \geq 0$ such that

$$\lim_{x\to\infty} h(x) = 0 \quad , \quad \lim_{x\to\infty} xh'(x) = 0$$

and

$$\sup_{t\geq t_0}\left|\frac{c''(t)c'(t)^{-\frac{1}{1+\gamma}}}{c'(t)h(c'(t))}\right| < \infty \qquad \text{for some } t_0 > 0.$$

Then the continuation region is given by

$$\mathcal{C} = \{(x,t) : |x| < \beta^*(t)\}$$

with

$$\beta^*(t) = (\frac{32c'(t)}{q})^{-\frac{1}{1+\gamma}}(1+o(1)) \quad . \tag{5.27}$$

Proof: As in the Brownian motion case we have to examine (R1)-(R3) and (cv1)-(cv6). Then we can apply Corollary (5.1.7), (5.1.8) which provide the assertion.

From the linear case we know that (R1)-(R3) hold; see Proposition 4.1.3. Furthermore the inverse function ψ of G'/U' fulfills

$$\psi(z) = (\frac{32z}{q})^{-\frac{1}{1+\gamma}}(1+o(1)) \quad \text{for } z \to \infty, \tag{5.28}$$

see Theorem 3.3.4. Since E is bounded and G is concave the conditions (cv1)-(cv3) are obviously fulfilled. We define the inner and outer approximation by

$$\zeta_+(t) = \psi(c'(t)) \quad , \quad \zeta_-(t) = \psi(c'(t)(1+h(c'(t))) \quad .$$

Then (5.28) and $c'(\infty) = \infty$ imply

$$\begin{aligned} \zeta_+(t) &= (\frac{32c'(t)}{q})^{-\frac{1}{1+\gamma}}(1+o(1)) \quad , \\ \zeta_-(t) &= (\frac{1}{q}32c'(t)(1+h(c'(t)))^{-\frac{1}{1+\gamma}}(1+o(1)). \end{aligned} \tag{5.29}$$

Thus both are asymptotically equivalent and it remains to examine condition (cv6) in this situation. Due to

$$\lim_{y\to 0} \frac{U(y)}{y} = 0$$

condition (cv6) is fulfilled if

$$\frac{c''(t)\zeta_+(t)}{c'(t)h(c'(t))}$$

remains bounded in t, and this holds if

$$\frac{c''(t)c'(t)^{-\frac{1}{1+\gamma}}}{c'(t)h(c'(t))}$$

is bounded for large t; see (5.28). □

We want to use the above result for some special reward functions.

1. $g(x) = |x - \frac{1}{2}|$:
   Then $G(y) = y, \gamma = 0, q = 1$, and the preceding theorem can be applied if we can find $h(x) = x^{-\delta}$ with $\delta > 0$ such that

$$\sup_{t\geq t_0}\left|\frac{c''(t)c'(t)^{-2}}{h(c'(t))}\right| = \sup_{t\geq t_0}\left|c''(t)c'(t)^{\delta-2}\right| < \infty \quad . \tag{5.30}$$

For the following cost functions we obtain a continuation region of the form

$$\mathcal{C} = \{(x,t) : |x - \frac{1}{2}| < \beta^*(t)\}$$

and we can determine the asymptotics of β^*.

1.1 $c(t) = t^\alpha$ with $\alpha > 1$:
Then we choose δ with $0 < \delta < \frac{\alpha}{\alpha-1}$ and obtain that

$$c''(t)c'(t)^{\delta-2} = O(t^{\alpha-2+(\alpha-1)(\delta-2)})$$

remains bounded in t since the exponent is less than zero. Hence

$$\beta^*(t) = (\frac{1}{32c'(t)})(1+o(1)) = \frac{1}{32\alpha}t^{-(\alpha-1)}(1+o(1)) \quad . \tag{5.31}$$

1.2 $c(t) = e^{\lambda t}$ with $\lambda > 0$:
Then we may use any δ with $0 < \delta < 1$, noting that

$$c''(t)c'(t)^{\delta-2} = O(e^{\lambda t(\delta-1)})$$

for $t \longrightarrow \infty$. Thus

$$\beta^*(t) = \frac{1}{32\lambda}e^{-\lambda t}(1+o(1)) \quad . \tag{5.32}$$

2. $g(x) = G(|x - \frac{1}{2}|)$ with $G(y) = -(\frac{1}{2} - y)^\nu$ for $\nu \geq 1$:
Then $G'(y) = \nu(\frac{1}{2} - y)^{\nu-1}$ tends to $q = \nu(\frac{1}{2})^{\nu-1}$. Thus, with $\gamma = 0$, we can apply Theorem 5.2.4 for the cost functions $c(t) = t^\alpha$ and $c(t) = e^{-\lambda t}$ as in the preceding example. We obtain

$$\beta^*(t) = \frac{1}{32\alpha q}t^{-(\alpha-1)}(1+o(1)) \tag{5.33}$$

in the first case and

$$\beta^*(t) = \frac{1}{32\lambda q}e^{-\lambda t}(1+o(1)) \tag{5.34}$$

in the second case.

5.2.5 Portfolio optimization

As was mentioned in the earlier chapters, see 2.2.3 or 2.3.3, a treatment of portfolio strategies without transaction costs leads to a diffusion with state space $E = (0, 1)$ and generator

$$A = \frac{1}{2}x^2(1-x)^2\partial_x^2 + x(1-x)(\frac{1}{2}-x)\partial_x \quad .$$

This is a mean reverting diffusion process symmetric w.r.t. $m = \frac{1}{2}$. The even solution of $Au = 1$, vanishing at $\frac{1}{2}$, is given by $u(x) = (\log(x/(1-x))^2$. Thus $U(y) = u(1/2 + y)$ fulfills

$$U(y) = (\log\frac{1/2+y}{1/2-y})^2 \quad , \quad U'(y) = 2\frac{\log\frac{1/2+y}{1/2-y}}{(1/2+y)(1/2-y)}$$

for all $y \in (0, 1/2)$.

We consider the same reward functions as in the concave case. We will determine how the continuation region shrinks.

1. $g(x) = |\log(\frac{x}{1-x})|$:
   As was seen in Chapter 4.2.5, $F = G'/U'$ fulfills

   $$F(y) = \frac{1}{2}(\log\frac{1/2+y}{1/2-y})^{-1} \quad , y \in (0, \frac{1}{2}),$$

   and has the inverse

   $$\psi(z) = \frac{1}{2}\frac{\exp(\frac{1}{2z})-1}{\exp(\frac{1}{2z})+1} = \frac{1}{4}\frac{1}{2z}(1+o(1)) \quad \text{for } z \to \infty. \tag{5.35}$$

   For a convex cost function c, we consider the inner and outer approximations

   $$\zeta_+(t) = \psi(c'(t)) \quad , \quad \zeta_-(t) = \psi(c'(t)(1 + h(c'(t))) \quad .$$

   Due to (5.35), they are asymptotically equivalent.

   To examine condition (cv6) we recall that $U(\psi(z)) = (\frac{1}{2z})^2$. Hence h must satisfy

   $$\lim_{t\to\infty}\frac{c''(t)}{h(c'(t))c'(t)^3} = 0$$

   which we met before in the Brownian motion case . Thus all examples for cost functions there may be carried over to this situation.

1.1 $c(t) = t^\alpha$ with $\alpha > 1$:
Then we choose $h(x) = x^{-\delta}$ with $0 < \delta < \frac{2\alpha-1}{\alpha-1}$ and obtain

$$\mathcal{C} = \{(x,t) : |x - \frac{1}{2}| < \beta^*(t)\}$$

with

$$\begin{aligned} \beta^*(t) &= \psi(c'(t))(1+o(1)) = \frac{1}{8c'(t)}(1+o(1)) \\ &= \frac{1}{8\alpha} t^{1-\alpha}(1+o(1)) \quad . \end{aligned} \tag{5.36}$$

1.2 $c(t) = e^{\lambda t}$ with $\lambda > 0$:
We choose $h(x) = x^{-\delta}$ with $0 < \delta < 2$ and obtain that the boundary of the continuation region satisfies

$$\beta^*(t) = \frac{1}{8\lambda} e^{-\lambda t}(1+o(1)) \quad . \tag{5.37}$$

2. $g(x) = \begin{cases} \log(1/(1-x)) & \text{,if } x > 1/2 \\ \log(1/x) & \text{,if } x \leq 1/2 \end{cases}$:
Then $F = G'/U'$ fulfills

$$F(y) = (\frac{1}{2} + y)\frac{1}{2\log(\frac{1/2+y}{1/2-y})} \tag{5.38}$$

which is strictly decreasing from infinity to zero. We denote by ψ its inverse which cannot be explicitly determined. We will show in the following that the inner and outer approximation

$$\zeta_+(t) = \psi(c'(t)) \quad , \quad \zeta_-(t) = \psi(c'(t)(1 + h(c'(t)))$$

are asymptotically equivalent. Since $F(y) \asymp (4\log\frac{1/2+y}{1/2-y})^{-1}$ for $y \to 0$ and $c'(t) = F(\zeta_+(t))$, $c'(t)(1+h(c'(t)) = F(\zeta_-(t))$ we have

$$\lim_{t\to\infty} \frac{\log(\frac{\frac{1}{2}+\zeta_+(t)}{\frac{1}{2}-\zeta_+(t)})}{\log(\frac{\frac{1}{2}+\zeta_-(t)}{\frac{1}{2}-\zeta_-(t)})} = 1 \quad . \tag{5.39}$$

Furthermore

$$F'(y) \asymp -(\log\frac{\frac{1}{2}+y}{\frac{1}{2}-y})^{-2} \qquad \text{for } y \to 0,$$

and therefore with $\eta(t) = 1 + h(c'(t))$, $f(t) = c'(t)\eta(t)$

$$\begin{aligned}
\lim_{t\to\infty} \frac{\zeta_+(t)}{\zeta_-(t)} &= \lim_{t\to\infty} \frac{\psi'(c'(t))c''(t)}{\psi'(c'(t)\eta(t))f'(t)} \\
&= \lim_{t\to\infty} \frac{F'(\psi(c'(t)\eta(t))}{F'(\psi(c'(t)))} \\
&= \lim_{t\to\infty} \frac{F'(\zeta_-(t))}{F'(\zeta_+(t))} \\
&= \lim_{t\to\infty} \frac{\log(\frac{\frac{1}{2}+\zeta_+(t)}{\frac{1}{2}-\zeta_+(t)})}{\log(\frac{\frac{1}{2}+\zeta_-(t)}{\frac{1}{2}-\zeta_-(t)})} \quad .
\end{aligned}$$

To examine (cv6) we use $U(y) \asymp \frac{1}{16F(y)^2}$ for $y \to 0$ and obtain that h satisfies (cv6) if

$$\frac{c''(t)}{c'(t)^3 h(c'(t))} \longrightarrow 0$$

is fulfilled. Hence the cost functions of the previous example may be used here too.

Chapter 6

Further extensions

So far we have formulated sufficient conditions that lead to continuation regions of the form

$$\mathcal{C} = \{(x,t) : |x-m| < \beta^*(t)\}$$

with a boundary function β^* which is increasing for concave costs of observations and decreasing in the convex case. We want to weaken our assumptions with the aim to apply the methods of the preceding chapters to a wider class of reward and cost functions.

6.1 The concave case

In the preceding chapters we obtained results concerning the asymptotic behaviour of the continuation region by supposing condition which partly had a non asymptotic structure, and thus can be improved.

Let us first consider the reward function $g(x) = G(|x-m|)$ for all $x \in E$. Since for concave observations the linear case has to be applied for small cost rates we replace (R1)-(R3) by the following conditions

(Rcc1) G is twice continuously differentiable and strictly increasing.

(Rcc2) $\frac{G'}{U'}$ is strictly decreasing with

$$\lim_{y\to 0} \frac{G'(y)}{U'(y)} = c_0 \quad , \quad \lim_{y\to l} \frac{G'(y)}{U'(y)} = 0 \quad .$$

(Rcc3) Ag is decreasing on $(m+l-\varepsilon, m+l)$ or $Ag \leq 0$ on $(m+l-\varepsilon, m+l)$ for some $\varepsilon > 0$.

Due to (Rcc2), the function $F(y) = G'(y)/U'(y)$ for all $y \in (0, l)$ has an inverse

$$\psi : (0, c_0) \to (0, l) \quad .$$

From (Rcc3), the results for the linear case can be applied for all cost rates $c \leq c_0$ such that $\psi(c) \geq l - \varepsilon$, hence on $(0, c_1)$ with $c_1 = F(l - \varepsilon)$.

Furthermore we examine for a cost function c how the conditions (cc1)-(cc6) can be modified to obtain an asymptotic result for the continuation region. The first three assumptions can be replaced in the following way since we only need concavity for large t.

There exists some $t_0 \geq 0$ such that

(Acc1) c is strictly increasing on (t_0, ∞) with $\lim_{t\to\infty} c(t) = \infty$,

(Acc2) c is twice continuously differentiable and concave on (t_0, ∞) with $\lim_{t\to\infty} c'(t) = 0$,

(Acc3) For each $x \in E$ there exists an $\alpha \in (0, 1)$ such that

$$E_x \sup_{t\geq 0}(g(X_t) - \alpha c(t)) < \infty \quad .$$

Then there exists some $t_1 > t_0$ such that $c'(t) < c_1$ for all $t > t_1$ and an inner approximation can be defined by

$$\beta_-(t) = \psi(c'(t)) \qquad \text{for all } t > t_1 \quad .$$

Since the conditions (cc1)-(cc3) are from an asymptotical nature we do not need to modify them but we have to keep in mind that the outer approximation is only defined by

$$\beta_+(t) = \psi(c'(t)(1 - h(c'(t))) \qquad \text{for all } t > t_1.$$

Then the same methods as in Chapter 4 work and we obtain the following result:

6.1.1 Theorem: *Let the reward and cost function have the properties (Rcc1)-(Rcc3), (Acc1)-(Acc3), (cc4)-(cc6). Then, with $t_1 > 0$ as above, the continuation region $\mathcal{C}$ has the form*

$$\mathcal{C} \cap (E \times (t_1, \infty)) = \{(x, t) : t > t_1, |x - m| < \beta^*(t)\}$$

with an increasing left continuous boundary function β^*. Furthermore

$$1 = \begin{cases} \lim_{t\to\infty} \frac{l-\beta^*(t)}{l-\beta_-(t)} & ,\textit{if } l < \infty \\ \lim_{t\to\infty} \frac{\beta^*(t)}{\beta_-(t)} & ,\textit{if } l = \infty \end{cases} \tag{6.1}$$

6.2 The convex case

An analogous argumentation as in the preceding section works for convex costs too. Here we keep in mind that the cost rate tends to infinity and the continuation region shrinks. Thus the function $F = G'/U'$ must tend to infinity for y tending to zero and the appropriate behaviour for Ag near m must be supposed. To be precise we formulate the following conditions.

(Rcv1) G is twice continuously differentiable and strictly increasing.

(Rcv2) $\frac{G'}{U'}$ is strictly decreasing with

$$\lim_{y\to 0} \frac{G'(y)}{U'(y)} = \infty \quad , \quad \lim_{y\to l} \frac{G'(y)}{U'(y)} = c_0 \quad .$$

(Rcv3) $Ag \leq 0$ or there exists some $\varepsilon > 0$ such that Ag is decreasing on $(m, m+\varepsilon)$ and $Ag(x) \leq Ag(m+\varepsilon)$ for all $x > m+\varepsilon$.

Due to (Rcv2), the function F has an inverse

$$\psi : (c_0, \infty) \to (0, l) \quad .$$

From (Rcv3), the linear case can be applied for all cost rates $c \geq c_0$ such that $\psi(c) \leq \varepsilon$, hence for all $c \geq c_1 = F(\varepsilon)$.

As before the cost function needs to be convex only for large t. Thus we suppose that there exists some $t_0 > 0$ such that

(Acv1) c is strictly increasing on (t_0, ∞) with $\lim_{t\to\infty} c(t) = \infty$,

(Acv2) c is twice continuously differentiable and convex on (t_0, ∞) with $\lim_{t\to\infty} c'(t) = \infty$,

(Acv3) For all $k > c_1$ and all $x \in E$

$$E_x \sup_{t\geq 0} (g(X_t) - kt) < \infty \quad .$$

Then for some $t_1 > t_0$ such that $c'(t_1) \geq c_1$ we can define the outer approximation $\beta_+(t) = \psi(c'(t))$. By using the properties (cv4)-(cv6) and keeping in mind that the inner approximation is defined for all $t > t_1$ through $\beta_-(t) = \psi(c'(t)(1 + h(c'(t)))$, we obtain the following result.

6.2.1 Theorem: *Let the reward and cost function have the properties (Rcv1)-(Rcv3), (Acv1)-(Acv3), (cv4)-(cv6). Then, with the precedingly defined t_1, the continuation region satisfies*

$$\mathcal{C} \cap (E \times (t_1, \infty)) = \{(x,t) : t > t_1, |x - m| < \beta^*(t)\}$$

with a decreasing right continuous boundary function β^*. Furthermore

$$\lim_{t\to\infty} \frac{\beta^*(t)}{\beta_+(t)} = 1 \quad . \tag{6.2}$$

6.3 Applications

As in Chapter 4.2.5, we consider the A-diffusion on $E = (0, 1)$ with generator

$$A = \frac{1}{2}x^2(1-x)^2\partial_x^2 + x(1-x)(\frac{1}{2} - x)\partial_x \quad ,$$

and we want to solve the optimal stopping problem for the reward function

$$(x,t) \to \log\frac{1}{1-x} - c(t)$$

for concave c. This problem with linear costs is investigated by Morton and Pliska [47] and has applications for portfolio optimization. Two problems arise. The first one is that $g(x) = \log\frac{1}{1-x}$ is not even w.r.t. $1/2$. Hence all results, so far obtained, cannot be applied. We can circumvent this by a symmetrization argument, introducing the reward function

$$g_1(x) = \frac{1}{2}(\log\frac{1}{1-x} + \log\frac{1}{x}) \quad .$$

For this reward the condition (R2) no longer holds, whereas the improved conditions of Theorem 6.1.1 can be examined. To be precise we first state the following reduction.

6.3.1 Proposition *The continuation regions for the reward functions $g(x) = \log\frac{1}{1-x}$ and $g_1(x) = \frac{1}{2}(\log\frac{1}{1-x} + \log\frac{1}{x})$ coincide.*

Proof: This follows from the fact that

$$g(x) = g_1(x) + \frac{1}{2}s(x) \qquad \text{for all } x \in (0,1)$$

with $s(x) = \log \frac{x}{1-x}$ denoting the scale function of the diffusion. Since $s(X_t)$ behaves like a Wiener process with starting point $s(x)$ w.r.t P_x , see Prop. 2.2.4,

$$E_x(s(X_\tau)) = s(x)$$

for each stopping time τ with $E_x\tau < \infty$. Therefore

$$E_x(g(X_\tau) - c(\tau)) = E_x(g_1(X_\tau) - c(\tau)) + \frac{1}{2}s(x) \quad .$$

Thus the optimization problems w.r.t. the reward functions g and g_1 are equivalent and the assertion is shown. □

Hence we analyse the stopping problem w.r.t. the symmetric reward function g_1 and have to examine the conditions (Rcc1)-(Rcc3), (Acc1)-(Acc3), (cc4)-(cc6) for a concave cost function c. We introduce

$$G_1(y) = g_1(\frac{1}{2} + y) = -\frac{1}{2}\log((\frac{1}{2} - y)(\frac{1}{2} + y))$$

for all $y \in (0, \frac{1}{2})$ and recall

$$U(y) = u(\frac{1}{2} + y) = (\log \frac{\frac{1}{2} + y}{\frac{1}{2} - y})^2 \quad , \quad U'(y) = 2\frac{\log \frac{\frac{1}{2}+y}{\frac{1}{2}-y}}{(\frac{1}{2} + y)(\frac{1}{2} - y)}.$$

Hence $F = G_1'/U'$ fulfills

$$F(y) = \frac{1}{2}\frac{y}{\log \frac{\frac{1}{2}+y}{\frac{1}{2}-y}} \qquad \text{for all } y \in (0, \frac{1}{2}).$$

Now $Ag_1(x) = \frac{1}{2}x(1-x)$ is decreasing on $(\frac{1}{2}, 1)$. Furthermore, F is strictly decreasing on $(0, 1/2)$ with $\lim_{y\to 0} F(y) = \frac{1}{8}$, $\lim_{y\to 1/2} F(y) = 0$. Thus the conditions (Rcc1)-(Rcc3) are valid whereas (R2) does not hold.

To obtain the asymptotics w.r.t. a concave cost function the same arguments as in 4.2.5 do work here. Let ψ denote the inverse of F which cannot be calculated explicitly. Therefore we introduce

$$F_2(y) = -\frac{1}{4\log(\frac{1}{2} - y)} \qquad \text{for all } y \in (0, \frac{1}{2})$$

and note that

$$\lim_{y\to\frac{1}{2}} \frac{F(y)}{F_2(y)} = 1 \quad .$$

The inverse of F_2, denoted by ψ_2, satisfies

$$\psi_2(z) = \frac{1}{2} - \exp(-\frac{1}{4z}) \quad , \quad \psi_2'(z) = -\frac{1}{4}\frac{1}{z^2}\exp(-\frac{1}{4z})$$

for all $z \in (0,\infty)$. We follow the procedure of the second example of 4.2.5 and note that it is easy to show

$$\lim_{y\to\frac{1}{2}} \psi_2'(F(y))F'(y) = 1 \quad .$$

This implies

$$\lim_{z\to 0} \frac{\frac{1}{2} - \psi_2(z)}{\frac{1}{2} - \psi(z)} = 1 \quad .$$

From this the asymptotic equivalence of the inner and outer approximation follows as in 4.2.5. Furthermore, condition (cc6) is valid if for a given c a function h can be choosen such that the condition

$$\lim_{t\to\infty} \frac{c''(t)}{c'(t)^3 h(c'(t))} = 0$$

is satisfied.

For example take $c(t) = t^\alpha$ with $\frac{2}{3} < \alpha < 1$. We may choose $h(x) = x^\delta$ with $1 < x < \frac{2\alpha-1}{1-\alpha}$ to obtain the following result for the continuation region $\mathcal{C}$: Setting $t_1 = (2\alpha)^{\frac{1}{1-\alpha}}$, $c'(t) < \frac{1}{2}$ for all $t > t_1$ and

$$\mathcal{C} \cap (E \times (t_1,\infty)) = \{(x,t) : t > t_1, |x - \frac{1}{2}| < \beta^*(t)\}$$

with an increasing function β^* satisfying

$$\begin{aligned} \frac{1}{2} - \beta^*(t) &= (\frac{1}{2} - \psi(c'(t)))(1+o(1)) \\ &= (\frac{1}{2} - \psi_2(c'(t)))(1+o(1) \\ &= \exp(-\frac{1}{4c'(t)})(1+o(1)) \quad . \end{aligned} \tag{6.3}$$

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