J. Appl. Prob. **34**, 1079–1082 (1997)
Printed in Israel

THE MOMENTS OF FIND

VOLKERT PAULSEN,* *Universität Kiel*

Abstract

To study the limiting behaviour of the random running-time of the FIND algorithm, the so-called FIND process was introduced by Grübel and Rösler [1]. In this paper an approach for determining the nth moment function is presented. Applied to the second moment this provides an explicit expression for the variance.

STOCHASTIC ALGORITHMS; FIXED-POINT METHOD

AMS 1991 SUBJECT CLASSIFICATION: PRIMARY 60E99
SECONDARY 60K99; 68P10

1. Introduction

The FIND algorithm, introduced by Hoare [3], is widely used to determine the kth smallest element of a set of n numbers. The asymptotic behaviour of the distribution of the number of comparisons needed by the algorithm was investigated by Grübel and Rösler [1], [2]. They introduced the limiting process $(Z_t)_{t\in[0,1]}$, the so-called FIND process. The marginal distributions of Z are uniquely defined by the following fixed-point equation:

$$Z_t \overset{\text{distr}}{=} 1 + UZ_{t/U}\mathbf{1}_{\{U>t\}} + (1-U)\bar{Z}_{(t-U)/(1-U)}\mathbf{1}_{\{U\leqq t\}}. \tag{1.1}$$

Here, Z, $\bar{Z}$ are independent processes with the same marginal distributions and U is uniformly distributed independent of them.

Equation (1.1) is the key for the proof of various properties of the FIND process Z. For example, Grübel and Rösler [1], [2] calculated from (1.1) the expectation $(EZ_t)_{0\leqq t\leqq 1}$.

In this paper we not only give an explicit formula for the second moment but also a procedure that provides the nth moment for $n\in\mathbb{N}$. Two main steps arise. In Section 2 we determine the fixed-point equation for the nth moment of Z_t. Proceeding from this we find a differential equation satisfied by a derivative of the nth moment function in Section 3. Finally, in Section 4 we apply the approach to the first and second moment and obtain explicit formulas for them.

Received 12 April 1996; revision received 13 January 1997.
* Postal address: Mathematisches Seminar, Universität Kiel, D-24098 Kiel, Germany.

2. Fixed-point equation for the nth moment function

Let

$$m_n(t) = EZ_t^n \qquad \text{for all } t \in [0, 1]$$

denote the nth moment function of the FIND process. Due to the fact that the distribution of each Z_t is majorized by a probability measure ν on the real line according to stochastic ordering, m_n is uniformly bounded by the nth moment of ν; see Grübel and Rösler [2].

We want to give a characterization of m_n as the unique fixed-point of some contraction. Let $\mathscr{F}_b$ be the space of bounded measurable functions on the unit interval. With the supremum norm $\mathscr{F}_b$ is a Banach space. For $n \in \mathbb{N}$, operators $K_n : \mathscr{F}_b \to \mathscr{F}_b$ are defined by

$$K_n(f)(t) = \int_t^1 u^n f\left(\frac{t}{u}\right) du + \int_0^t (1-u)^n f\left(\frac{t-u}{1-u}\right) du + b_n(t), \tag{2.1}$$

where

$$b_n(t) = \sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i} m_{n-i}(t). \tag{2.2}$$

Then the following holds.

Theorem 2.1. *K_n is a strict contraction in the supremum norm and m_n is the unique fixed-point of K_n for each $n \in \mathbb{N}$. $(K_n^k(f))_{k \in \mathbb{N}}$ converges exponentially fast to m_n in the supremum norm for each $f \in \mathscr{F}_b$.*

Proof. For $f, g \in \mathscr{F}_b$

$$\begin{aligned}
&\|K_n f - K_n g\|_\infty \\
&= \sup_{t \in [0,1]} \left| \int_t^1 u^n \left(f\left(\frac{t}{u}\right) - g\left(\frac{t}{u}\right)\right) du + \int_0^t (1-u)^n \left(f\left(\frac{t-u}{1-u}\right) - g\left(\frac{t-u}{1-u}\right)\right) du \right| \\
&\leqq \sup_{t \in [0,1]} \left(\int_t^1 u^n du + \int_0^t (1-u)^n du \right) \|f-g\|_\infty \\
&\leqq \frac{1}{n+1} (2 - (\tfrac{1}{2})^n) \|f-g\|_\infty .
\end{aligned}$$

Thus K_n is a contraction in the supremum norm and the fixed-point theorem of Banach provides exponentially fast convergence of $(K_n^n(f))_{k \in \mathbb{N}}$ to the unique fixed-point of K_n for each starting point f.

It remains to show the fixed-point property of m_n, i.e.

$$K_n(m_n) = m_n \qquad \text{for all } n \in \mathbb{N}. \tag{2.3}$$

From (1.1) we obtain

$$E(Z_t-1)^n=\int_t^1 u^n m_n\left(\frac{t}{u}\right)du+\int_0^t (1-u)^n m_n\left(\frac{t-u}{1-u}\right)du. \tag{2.4}$$

Hence (2.3) follows from

$$E(Z_t-1)^n=m_n(t)+\sum_{i=1}^{n}\binom{n}{i}(-1)^i m_{n-i}(t).$$

The contraction K_n, defined in (2.1), is an affine operator, which can be written as $K_n=A_n+b_n$ with

$$A_n f(t)=\int_t^1 u^n f\left(\frac{t}{u}\right)du+\int_0^t (1-u)^n f\left(\frac{t-u}{1-u}\right)du \quad \text{for all } f\in\mathscr{F}_b. \tag{2.5}$$

In terms of the linear part A_n we obtain the following representation of the nth moment function m_n.

Corollary 2.2. $m_n=\Sigma_{k=0}^{\infty} A_n^k b_n$.

Proof. Starting with the function b_n, Theorem 2.1 implies $m_n=\lim_{k\to\infty} K_n^k(b_n)$. By induction we obtain $K_n^k(b_n)=\Sigma_{j=0}^{k} A_n^j b_n$, from which the assertion follows.

3. Differential equation for the nth moment function

After substitution the fixed-point property (2.3) of m_n can be written in the form

$$m_n(t)=t^{n+1}\int_t^1 \frac{m_n(s)}{s^{n+2}}ds+(1-t)^{n+1}\int_{1-t}^1 \frac{m_n(1-s)}{s^{n+2}}ds+b_n(t). \tag{3.1}$$

Thus it is rather obvious that m_n is an infinitely often differentiable function. To calculate the derivatives let us define the operator ψ_n by

$$\psi_n(f)(t)=t^n\int_t^1 \frac{f(s)}{s^{n+1}}ds \qquad \text{for all } f\in C([0,1]). \tag{3.2}$$

Using the Leibnitz formula for differentiation of products and taking into account

$$D^k t^n=\frac{1}{t}\frac{n-k+1}{n+1}D^k t^{n+1}, \qquad 0\leqq k\leqq n$$

it is easy to see that

$$D^{n+1}\psi_n(f)(t)=-\frac{1}{t}D^n f(t). \tag{3.3}$$

Here D denotes the derivative operator.

An analogous treatment of the second integral in (3.1) leads to

$$D^{n+2}m_n(t) = \left(\frac{1}{1-t} - \frac{1}{t}\right) D^{n+1}m_n(t) + D^{n+2}b_n(t). \tag{3.4}$$

Equation (3.4) implies that the $(n+1)$th derivative of the nth moment function is a solution to an ordinary linear differential equation, which can easily be solved. Thus we have a procedure to determine recursively the nth moment function.

4. Explicit solution for the variance

In the case $n=1$ the differential equation (3.4) immediately leads to the known formula of the expectation

$$m_1(t) = 2 - 2t\ln t - 2(1-t)\ln(1-t) \qquad \text{for all } t \in [0, 1]. \tag{4.1}$$

Proceeding from this we obtain the third derivative of the second moment function by solving the differential equation

$$y' = \left(-\frac{1}{t} + \frac{1}{1-t}\right)y - 8\left(\frac{1}{t^3} + \frac{1}{(1-t)^3}\right), \qquad y(\tfrac{1}{2}) = 0.$$

Integration and insertion into the fixed-point equation (3.1) finally leads to

$$\begin{aligned} m_2(t) &= \tfrac{9}{2} + 5t(1-t) - 12(t\ln t + (1-t)\ln(1-t)) \\ &\quad + 2(t^2\ln^2 t + (1-t)^2\ln^2(1-t)) \\ &\quad + 4(t^2(\operatorname{dilog}(1-t) - \tfrac{1}{6}\pi^2) + (1-t)^2(\operatorname{dilog}(t) - \tfrac{1}{6}\pi^2)), \end{aligned} \tag{4.2}$$

with $\operatorname{dilog}(t) = \int_1^t (\ln y)/(1-y)dy$ for all $t > 0$.

From the above formulas for the expectation and second moment we can immediately obtain the following expression for the variance:

$$\begin{aligned} \operatorname{Var}(Z_t) &= \tfrac{1}{2} + 5t(1-t) - 4(t\ln t + (1-t)\ln(1-t)) - 2(t^2\ln^2 t + (1-t)^2\ln^2(1-t)) \\ &\quad - 8t(1-t)\ln(t)\ln(1-t) + 4(t^2(\operatorname{dilog}(1-t) - \tfrac{1}{6}\pi^2) + (1-t)^2(\operatorname{dilog}(t) - \tfrac{1}{6}\pi^2)). \end{aligned}$$

Acknowledgement

The author wishes to thank the referee for his helpful comments.

References

[1] GRÜBEL, R. AND RÖSLER, U. (1996) Asymptotic distribution theory for Hoare's selection algorithm. *Adv. Appl. Prob.* **28**, 252–269.

[2] GRÜBEL, R. AND RÖSLER, U. (1995) The backward view on the algorithm FIND. *Preprint.*

[3] HOARE, C. A. R. (1961) Algorithm 63, Partition; Algorithm 64, Quicksort; Algorithm 65, Find. *Commun. ACM* **4**, 321–322.