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THE MOMENTS OF FIND

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Abstract

To study the limiting behaviour of the random running-time of the FIND algorithm, the so-called FIND process was introduced by Grübel and Rösler [1]. In this paper an approach for determining the n th moment function is presented. Applied to the second moment this provides an explicit expression for the variance.

STOCHASTIC ALGORITHMS; FIXED-POINT METHOD

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1. Introduction

The FIND algorithm, introduced by Hoare [3], is widely used to determine the k th smallest element of a set of n numbers. The asymptotic behaviour of the distribution of the number of comparisons needed by the algorithm was investigated by Grübel and Rösler [1], [2]. They introduced the limiting process $(Z_t)_{t \in [0,1]}$, the so-called FIND process. The marginal distributions of Z are uniquely defined by the following fixed-point equation:

$$(1.1) \quad Z_t^{\text{distr}} = 1 + UZ_{t/U} \mathbf{1}_{\{U > t\}} + (1 - U)\tilde{Z}_{(t-U)/(1-U)} \mathbf{1}_{\{U \leq t\}}.$$

Here, Z , \tilde{Z} are independent processes with the same marginal distributions and U is uniformly distributed independent of them.

Equation (1.1) is the key for the proof of various properties of the FIND process Z . For example, Grübel and Rösler [1], [2] calculated from (1.1) the expectation $(EZ_t)_{0 \leq t \leq 1}$.

In this paper we not only give an explicit formula for the second moment but also a procedure that provides the n th moment for $n \in \mathbb{N}$. Two main steps arise. In Section 2 we determine the fixed-point equation for the n th moment of Z_t . Proceeding from this we find a differential equation satisfied by a derivative of the n th moment function in Section 3. Finally, in Section 4 we apply the approach to the first and second moment and obtain explicit formulas for them.

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2. Fixed-point equation for the n th moment function

Let

$$m_n(t) = EZ_t^n \quad \text{for all } t \in [0, 1]$$

denote the n th moment function of the FIND process. Due to the fact that the distribution of each Z_t is majorized by a probability measure ν on the real line according to stochastic ordering, m_n is uniformly bounded by the n th moment of ν ; see Grübel and Rösler [2].

We want to give a characterization of m_n as the unique fixed-point of some contraction. Let \mathcal{F}_b be the space of bounded measurable functions on the unit interval. With the supremum norm \mathcal{F}_b is a Banach space. For $n \in \mathbb{N}$, operators $K_n : \mathcal{F}_b \rightarrow \mathcal{F}_b$ are defined by

$$(2.1) \quad K_n(f)(t) = \int_t^1 u^n f\left(\frac{t}{u}\right) du + \int_0^t (1-u)^n f\left(\frac{t-u}{1-u}\right) du + b_n(t),$$

where

$$(2.2) \quad b_n(t) = \sum_{i=1}^n (-1)^{i-1} \binom{n}{i} m_{n-i}(t).$$

Then the following holds.

Theorem 2.1. *K_n is a strict contraction in the supremum norm and m_n is the unique fixed-point of K_n for each $n \in \mathbb{N}$. $(K_n^k(f))_{k \in \mathbb{N}}$ converges exponentially fast to m_n in the supremum norm for each $f \in \mathcal{F}_b$.*

Proof. For $f, g \in \mathcal{F}_b$

$$\begin{aligned} & \|K_n f - K_n g\|_\infty \\ &= \sup_{t \in [0,1]} \left| \int_t^1 u^n \left(f\left(\frac{t}{u}\right) - g\left(\frac{t}{u}\right) \right) du + \int_0^t (1-u)^n \left(f\left(\frac{t-u}{1-u}\right) - g\left(\frac{t-u}{1-u}\right) \right) du \right| \\ &\leq \sup_{t \in [0,1]} \left(\int_t^1 u^n du + \int_0^t (1-u)^n du \right) \|f - g\|_\infty \\ &\leq \frac{1}{n+1} (2 - (\tfrac{1}{2})^n) \|f - g\|_\infty. \end{aligned}$$

Thus K_n is a contraction in the supremum norm and the fixed-point theorem of Banach provides exponentially fast convergence of $(K_n^k(f))_{k \in \mathbb{N}}$ to the unique fixed-point of K_n for each starting point f .

It remains to show the fixed-point property of m_n , i.e.

$$(2.3) \quad K_n(m_n) = m_n \quad \text{for all } n \in \mathbb{N}.$$

From (1.1) we obtain

$$(2.4) \quad E(Z_t - 1)^n = \int_t^1 u^n m_n\left(\frac{t}{u}\right) du + \int_0^t (1-u)^n m_n\left(\frac{t-u}{1-u}\right) du.$$

Hence (2.3) follows from

$$E(Z_t - 1)^n = m_n(t) + \sum_{i=1}^n \binom{n}{i} (-1)^i m_{n-i}(t).$$

The contraction K_n , defined in (2.1), is an affine operator, which can be written as $K_n = A_n + b_n$ with

$$(2.5) \quad A_n f(t) = \int_t^1 u^n f\left(\frac{t}{u}\right) du + \int_0^t (1-u)^n f\left(\frac{t-u}{1-u}\right) du \quad \text{for all } f \in \mathcal{F}_b.$$

In terms of the linear part A_n we obtain the following representation of the n th moment function m_n .

Corollary 2.2. $m_n = \sum_{k=0}^{\infty} A_n^k b_n$.

Proof. Starting with the function b_n , Theorem 2.1 implies $m_n = \lim_{k \rightarrow \infty} K_n^k(b_n)$. By induction we obtain $K_n^k(b_n) = \sum_{j=0}^k A_n^j b_n$, from which the assertion follows.

3. Differential equation for the n th moment function

After substitution the fixed-point property (2.3) of m_n can be written in the form

$$(3.1) \quad m_n(t) = t^{n+1} \int_t^1 \frac{m_n(s)}{s^{n+2}} ds + (1-t)^{n+1} \int_{1-t}^1 \frac{m_n(1-s)}{s^{n+2}} ds + b_n(t).$$

Thus it is rather obvious that m_n is an infinitely often differentiable function. To calculate the derivatives let us define the operator ψ_n by

$$(3.2) \quad \psi_n(f)(t) = t^n \int_t^1 \frac{f(s)}{s^{n+1}} ds \quad \text{for all } f \in C([0, 1]).$$

Using the Leibnitz formula for differentiation of products and taking into account

$$D^k t^n = \frac{1}{t} \frac{n-k+1}{n+1} D^k t^{n+1}, \quad 0 \leq k \leq n$$

it is easy to see that

$$(3.3) \quad D^{n+1} \psi_n(f)(t) = -\frac{1}{t} D^n f(t).$$

Here D denotes the derivative operator.

An analogous treatment of the second integral in (3.1) leads to

$$(3.4) \quad D^{n+2}m_n(t) = \left(\frac{1}{1-t} - \frac{1}{t}\right) D^{n+1}m_n(t) + D^{n+2}b_n(t).$$

Equation (3.4) implies that the $(n+1)$ th derivative of the n th moment function is a solution to an ordinary linear differential equation, which can easily be solved. Thus we have a procedure to determine recursively the n th moment function.

4. Explicit solution for the variance

In the case $n=1$ the differential equation (3.4) immediately leads to the known formula of the expectation

$$(4.1) \quad m_1(t) = 2 - 2t \ln t - 2(1-t) \ln(1-t) \quad \text{for all } t \in [0, 1].$$

Proceeding from this we obtain the third derivative of the second moment function by solving the differential equation

$$y' = \left(-\frac{1}{t} + \frac{1}{1-t}\right)y - 8\left(\frac{1}{t^3} + \frac{1}{(1-t)^3}\right), \quad y(\tfrac{1}{2}) = 0.$$

Integration and insertion into the fixed-point equation (3.1) finally leads to

$$(4.2) \quad \begin{aligned} m_2(t) = & \frac{9}{2} + 5t(1-t) - 12(t \ln t + (1-t) \ln(1-t)) \\ & + 2(t^2 \ln^2 t + (1-t)^2 \ln^2(1-t)) \\ & + 4(t^2(\operatorname{dilog}(1-t) - \tfrac{1}{6}\pi^2) + (1-t)^2(\operatorname{dilog}(t) - \tfrac{1}{6}\pi^2)), \end{aligned}$$

with $\operatorname{dilog}(t) = \int_1^t (\ln y)/(1-y) dy$ for all $t > 0$.

From the above formulas for the expectation and second moment we can immediately obtain the following expression for the variance:

$$\begin{aligned} \operatorname{Var}(Z_t) = & \tfrac{1}{2} + 5t(1-t) - 4(t \ln t + (1-t) \ln(1-t)) - 2(t^2 \ln^2 t + (1-t)^2 \ln^2(1-t)) \\ & - 8t(1-t) \ln(t) \ln(1-t) + 4(t^2(\operatorname{dilog}(1-t) - \tfrac{1}{6}\pi^2) + (1-t)^2(\operatorname{dilog}(t) - \tfrac{1}{6}\pi^2)). \end{aligned}$$

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