Poroelasticity (linearized theory)

Poroelastic theory describes the behaviour of a porous elastic structure (the matrix material) immersed in fluid. It was developed to describe consolidation of soil, but can also be applied to biological tissues or rocks with porosity. First works by Karl von Terzaghi (~1925) and Maurice Biot (~1941, Columbia University).

kinematic variables : displacement u, volume fraction of fluid $\Delta\zeta$ (relative to reference state, i.e. $\Delta\zeta = \zeta - \zeta_0$)

dynamic variables : ~~stress of~~ total stress T, fluid pressure p
body force $\hat{b} = \frac{b}{\rho_R}$ and surface loads acting on elastic matrix

We directly look at linearized theory, i.e. assume small displacement u and ∇u as well as small changes in $\Delta\zeta$; also we assume isotropic materials and an incompressible fluid. Finally, here only statics. (Of course, all this can be generalized.)

Linearized theory $\Rightarrow$ reference state = relaxed porous matrix
Assume there exists an energy density of the poroelastic material, $W(\nabla u, \Delta\zeta)$

$$W(\nabla u, \Delta\zeta) \underset{\substack{\uparrow \\ \text{small } \nabla u,\ \text{relaxed reference state}}}{=} \tfrac{1}{2} W_{,11}(0,0)(\nabla u, \nabla u) + \tfrac{1}{2} W_{,22}(0,0)(\Delta\zeta, \Delta\zeta) + W_{,12}(0,0)(\nabla u, \Delta\zeta)$$

Balance laws : mass conservation of solid $\rightsquigarrow$ trivial, since linearized equations considered
lin. momentum conserv.: $\operatorname{div} T + \hat{b} = 0$ in Ω (A)
ang. moment. conserv.: T symmetric
mass conservation of fluid: $\frac{\partial}{\partial t}\Delta\zeta + \operatorname{div} f = 0$ in Ω (B)

for fluid volume flux per normal cross-sectional area f

(indeed, $0 \underset{\substack{\uparrow \\ \text{incompressibility}}}{=} \frac{d}{dt}\int_E \Delta\zeta \, dx + \int_{\partial E} f \cdot n \, da = \int_E \frac{\partial \Delta\zeta}{\partial t} + \operatorname{div} f \, dx$)

Boundary conditions:
$$\left.\begin{aligned} Tn &= s \text{ on } \partial\Omega_2 \\ u &= u_0 \text{ on } \partial\Omega_1 \end{aligned}\right\} \partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$$
$$\left.\begin{aligned} p &= p_0 \text{ on } \partial\Omega_3 \\ \frac{\partial p}{\partial n} &= \frac{f_0}{k} \text{ on } \partial\Omega_4 \end{aligned}\right\} \partial\Omega = \partial\Omega_3 \cup \partial\Omega_4$$

Constitutive laws: Gibbs free energy (btw, Gibbs lived also not too far away – he's interred on Yale cemetery)

$$E[u, \Delta\zeta] = \int_\Omega \underbrace{W(\nabla u, \Delta\zeta)}_{\text{stored energy}} - \underbrace{\hat{b} \cdot u}_{\text{potential of body force}} - \underbrace{p \cdot \Delta\zeta}_{\text{potential of pressure}} \, dx - \underbrace{\int_{\partial\Omega} s \cdot u \, da}_{\text{potential of surface load}}$$

in equil. $0 = \partial_{(u,\Delta\zeta)} E[u, \Delta\zeta](\vartheta, \xi) = \int_\Omega W_{,11}(0,0)\nabla u : \nabla\vartheta + W_{,22}(0,0)\Delta\zeta\,\xi + W_{,12}(0,0)(\nabla u, \xi) + W_{,12}(0,0)(\nabla\vartheta, \xi)$
$\qquad - \hat{b}\cdot\vartheta - p\,\xi \, dx - \int_{\partial\Omega} s \cdot \vartheta \, da$

$= \int_\Omega -[\operatorname{div}(W_{,11}(0,0)\nabla u + W_{,21}(0,0)\Delta\zeta) + \hat{b}] \cdot \vartheta \, dx + \int_{\partial\Omega} [(W_{,11}(0,0)\nabla u + W_{,21}(0,0)\Delta\zeta)n - s] \cdot \vartheta$
$\quad + \int_\Omega (W_{,22}(0,0)\Delta\zeta + W_{,12}(0,0)\nabla u ~~-p~~)\,\xi \, dx$

We readily identify
$$T = W_{,11}(0,0)\nabla u + W_{,21}(0,0)\Delta\zeta = \mathbb{K}\nabla u + \Delta\zeta H \qquad (a)$$
$$p = W_{,22}(0,0)\Delta\zeta + W_{,12}(0,0)\nabla u = g\,\Delta\zeta + H^T : \nabla u \qquad (b)$$

in addition: Darcy's law for porous flow: $f = -k\nabla p$ for permeability k (C)

isotropy implies:
- shear to the left and to the right produce same pressure change
  $\Rightarrow H_{12} = -H_{12} \Leftrightarrow H_{12} = 0$
  analogously, $H_{ij} = 0$ for $i \neq j$
- also, due to isotropy, $H_{11} = H_{22} = H_{33} =: \alpha g$ for "Biot-Willis parameter" α
- $\mathbb{K}$ satisfies same symmetries as normal elasticity tensor

Typically, constitutive laws are written in the form
$$T = \mathbb{C}\,\varepsilon(u) + d_1 p$$
$$\Delta\zeta = d_2 : \varepsilon(u) + d_3 p$$

we identify the coefficients $d_1, d_2 \in \mathbb{R}^{3\times 3}, d_3 \in \mathbb{R}$: (16)

$(b) \Rightarrow \Delta G = \frac{p}{g} + \alpha \operatorname{tr} \varepsilon(u)$, this in $(a) \Rightarrow T = \underbrace{\mathcal{K}\varepsilon(u) + \alpha^2 \operatorname{tr}(\varepsilon(u)) I}_{\text{stress due to deformation of solid matrix}} + \underbrace{\alpha p I}_{\text{pressure contribution to the total stress}}$

The last equation, the decomposition of the stress in a part due to the solid matrix and a pressure component, is called "Terzaghi's stress principle". The typically used coefficients are

$$T = 2\mu\, \varepsilon(u) + \lambda \operatorname{tr}(\varepsilon(u)) I \quad + \alpha p I \tag{D}$$

$$\Delta G = \alpha \operatorname{tr} \varepsilon(u) \quad + \frac{(1-\alpha\beta)\alpha}{K\beta} p \tag{E}$$

for Biot-Willis parameter α, Skempton's coefficient β, bulk modulus K and Lamé const. μ, λ of the matrix structure (not of the composing material!)

Rk: Solving the equations (A), (B), (C), (D), (E), one can see various interesting phenomena, e.g. that a porous solid only deforms slowly, when a load is applied, since all the fluid has to be squeezed out.

A list of further interesting topics in elasticity (still active research)

- dimension reduction (describe a 3D-solid of thickness h and find out what equations govern the deformation as $h \downarrow 0$) and resulting plate theory (or similar) (e.g. Friesecke, James, Müller: Comm. Pure Appl. Math. 2002)
- shape optimization (find the optimal geometry of an elastically loaded device) (e.g. Penzler, Rumpf, Wirth: ESAIM COCV 2012)
- micro-pattern formation in elastic objects (e.g. Bella, Kohn: Comm. Pure Appl. Math. 2013)
- phase transformations and martensites (Bhattacharya: "Microstructure of Martensite" 2003)

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