<u>Thm (Existence)</u>: Assume C to be elliptic, i.e. $(C\varepsilon):\varepsilon \geq \tilde{c}|\varepsilon|^2$ for a $\tilde{c}>0$ and $\forall \varepsilon \in \mathbb{R}^{3\times 3}_{sym}$.
Let $u_0 \in H^1(\Omega)$, $s \in L^2(\partial\Omega_2)$, $\hat{b} \in H^{-1}(\Omega)$.
Then $(*)$ has a weak solution $u \in H^1(\Omega)$.

proof: Apply the direct method to energy $E[u]$ to show it has a minimizer (homework). □

<u>Thm (Uniqueness)</u>: The solution to $(*)$ is unique.

proof: Let u_1, u_2 be solutions, then $\tilde{u} = u_1 - u_2$ solves the homogeneous problem $(*)$ with $s=0$, $\hat{b}=0$, $u_0=0$. Hence, $\operatorname{div} C\varepsilon(\tilde{u})=0$. Test with $\tilde{u}$ to obtain
$$0 = \int_\Omega \tilde{u}\cdot \operatorname{div} C\varepsilon(\tilde{u})\,dx = -\int_\Omega C\varepsilon(\tilde{u}):\varepsilon(\tilde{u})\,dx \geq -\tilde{c}\,\|\varepsilon(\tilde{u})\|^2_{L^2} \underset{\text{Korn}}{\geq} c\,\|\tilde{u}\|^2_{H^1} \Rightarrow \tilde{u}=0. \;\square$$

<u>Rk: For uniqueness, we basically just used the strict convexity of E.</u>

<u>Convex duality</u>

Legendre-Fenchel-dual or convex conjugate of a fcn $f: X \to \mathbb{R}\cup\{\infty\}$: $f^*: X^* \to \mathbb{R}\cup\{\infty\}$,
$$f^*(z) = \sup_{x\in X} \langle x,z\rangle - f(x)$$

biconjugate: $f^{**}: X \to \mathbb{R}\cup\{\infty\}$, $f^{**} = \sup_{z\in X^*} \langle x,z\rangle - f^*(z)$

properties:
- Fenchel inequality $f(x)+f^*(z) \geq \langle x,z\rangle \quad \forall (x,z)\in X\times X^*$ (from definition)
- f^* is convex lower semi-continuous
  (lsc, since it is ptwise sup of continuous fcns; convex, since
  $f^*(\theta z_1 + (1-\theta)z_2) = \sup_{x\in X} \langle x, \theta z_1 + (1-\theta)z_2\rangle - f(x) \leq \theta\left(\sup_{x\in X}\langle x,z_1\rangle - f(x)\right) + (1-\theta)\left(\sup_{x\in X}\langle x,z_2\rangle - f(x)\right)$)
- $f^{**}(x) = \sup_{z\in X^*} \langle x,z\rangle - \sup_{y\in X}\left(\langle y,z\rangle - f(y)\right) = \sup_{z\in X^*}\inf_{y\in X} \langle x-y,z\rangle + f(y) \leq f(x)$
- Fenchel-Moreau Thm: If f is proper, then f is lsc, convex $\Leftrightarrow f = f^{**}$

elastic energy density: $w(\varepsilon)$ (convex, differentiable; $\frac{1}{2}(C\varepsilon):\varepsilon$ for lin. elast.)

"elast. energy density of stress": $w^*(T)$ ($\frac{1}{2}(C^{-1}T):T$ for lin. elast.)

$\Rightarrow w(\varepsilon) + w^*(T) \geq \varepsilon : T$ by Fenchel inequality

<u>Lemma</u>: $T = w_{,\varepsilon}(\varepsilon)$ (stress-strain-law) $\Leftrightarrow w(\varepsilon) + w^*(T) = \varepsilon : T$

proof: "$\Rightarrow$": $\tilde{\varepsilon} \mapsto \tilde{\varepsilon}:T - w(\tilde{\varepsilon})$ is concave $\Rightarrow$ reaches its max at $\varepsilon \Rightarrow w^*(T) = \varepsilon:T - w(\varepsilon)$.
"$\Leftarrow$": we have $w^*(T) = \varepsilon:T - w(\varepsilon)$ which means that ε maximizes $\varepsilon:T - w(\varepsilon) \Rightarrow T = w_{,\varepsilon}(\varepsilon)$. □

Define: stored energy of strain, $\mathcal{W}[\varepsilon] = \int_\Omega w(\varepsilon)\,dx$,
stored energy of stress, $\mathcal{W}^*[T] = \int_\Omega w^*(T)\,dx$,
$$\langle \varepsilon, T\rangle = \int_\Omega \varepsilon : T\,dx.$$

Fenchel ineq. & Lemma imply: $\mathcal{W}[\varepsilon] + \mathcal{W}^*[T] \geq \langle\varepsilon,T\rangle$ with equality, iff $T = w_{,\varepsilon}(\varepsilon)$ a.e. (o)

Define:
- $u:\Omega\to\mathbb{R}^3$ is "kinematically admissible" if $u=u_0$ on $\partial\Omega_1$
- $T:\Omega\to\mathbb{R}^{3\times 3}_{sym}$ is "statically admissible" if $\operatorname{div} T + \hat{b} = 0$ in Ω and $Tn = s$ on $\partial\Omega_2$
- free energy of a kin. adm. u, $E[u] = \frac{1}{2}\mathcal{W}[\varepsilon(u)] \underbrace{- \int_\Omega \hat{b}\cdot u\,dx - \int_{\partial\Omega_2} s\cdot u\,da}_{-L[u]}$
- free energy of a stat. adm. T, $E^*[T] = -\frac{1}{2}\mathcal{W}^*[T] + \underbrace{\int_{\partial\Omega_1} Tn\cdot u_0\,da}_{K[T]}$

<u>Thm</u>: Let u and T be kin. adm. and stat. adm., resp., then
$$L[u] + K[T] = \langle \varepsilon(u), T\rangle \quad \text{and}$$
$$E^*[T] \leq E[u] \quad \text{with equality iff } u \text{ solves } (*) \text{ and } T \text{ is corr. stress.}$$

proof: $L[u]+K[T] = \int_\Omega \hat{b}\cdot u\,dx + \int_{\partial\Omega} Tn\cdot u\,da = -\int_\Omega \operatorname{div} T\cdot u\,dx + \int_{\partial\Omega} Tn\cdot u\,da = \int_\Omega T:Du\,dx = \langle\varepsilon(u),T\rangle$.
This together with (o) implies $E^*[T] \leq E[u]$ with equality iff T is the stress to u.
But this implies that u minimizes E and thus solves $(*)$. □

Rk: The previous thm. implies that one can solve either of the equivalent dual problems:
- minimize $E[u]$ among all kinem. adm. displacements u
- maximize $E^*[T]$ among all static. adm. stress fields T

Numerical solution via Finite Elements (FE)

Let $H^1_{u_0} = \{u \in H^1(\Omega) \mid u = u_0 \text{ on } \partial\Omega_1\}$

- Weak formulation of (*) : Find $u \in H^1_{u_0}$ s.t. $0 = \int_\Omega C\varepsilon(u):\varepsilon(\varphi) - \hat{b}\cdot\varphi \, dx - \int_{\partial\Omega_2} s\cdot\varphi \, da \quad \forall \varphi \in H^1_{0|_{\partial\Omega_1}}$.
- Let Ω be triangulated with nodes $\hat{x}_i \in \Omega$, $i = 1, \dots, N$, and regular triangles $T_j = (\hat{x}_{i_{j1}}, \hat{x}_{i_{j2}}, \hat{x}_{i_{j3}}) \subset \Omega$ of grid size h, let $V^h = \{u \in H^1(\Omega) \mid u \text{ is continuous}, u \text{ is affine on each } T_j\} \subset H^1(\Omega)$,
$$V^h_{u_0} = V^h \cap H^1_{u_0} \quad \text{be the spaces of finite element functions.}$$
For simplicity assume $u_0 \in V^h$, i.e. it is compatible with the triangulation.
- Note : $V^h = \text{span}\{\varphi_i^j \mid i = 1, \dots, N; j = 1,2,3\}$ where the $\varphi_i^j \in V^h$ are the FE Basis functions with $\varphi_i^j(\hat{x}_n) = \delta_{in} e_j$

- FE formulation: Find $u_h \in V^h_{u_0}$ s.t. $0 = \int_\Omega C\varepsilon(u_h):\varepsilon(\varphi_h) - \hat{b}\cdot\varphi_h \, dx - \int_{\partial\Omega_2} s\cdot\varphi_h \, da \quad \forall \varphi_h \in V^h_{0|_{\partial\Omega_1}}$ (**)
- Existence and uniqueness follow exactly as before by variational principle.
- Implementation: Can write $u_h = \sum_{i=1}^N \sum_{j=1}^3 U_i^j \varphi_i^j$, $U = (U_1^1, \dots, U_N^1, U_1^2, \dots, U_N^2, U_1^3, \dots, U_N^3)^T$

  Introduce stiffness matrices $L^{jl} = \left(\int_\Omega C\varepsilon(\varphi_i^j):\varepsilon(\varphi_k^l) \, dx\right)_{ik} \in \mathbb{R}^{N\times N}$

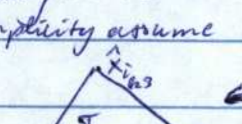
  and $L = \begin{pmatrix} L^{11} & L^{12} & L^{13} \\ L^{21} & L^{22} & L^{23} \\ L^{31} & L^{32} & L^{33} \end{pmatrix}$

  and mass vectors $B^j = \left(\int_\Omega \hat{b}\cdot\varphi_i^j \, dx + \int_{\partial\Omega_2} s\cdot\varphi_i^j \, da\right)_i$, $B = \begin{pmatrix} B^1 \\ B^2 \\ B^3 \end{pmatrix}$

  then (**) $\iff LU = B$ and $U_i^j = (u_0(\hat{x}_i))_j$ for all $\hat{x}_i \in \partial\Omega_1$

  $\iff \tilde{L}U = \tilde{B} \iff U = \tilde{L}^{-1}\tilde{B}$ where $\tilde{L}$ is L with all rows ~~and columns~~ belonging to the $\hat{x}_i \in \partial\Omega_1$ replaced by identity rows ~~and columns~~, and $\tilde{B}$ is B with the corresponding entries replaced by $(u_0(\hat{x}_i))_j$.

- Evaluating the matrices & vectors (illustration in 2D, 3D analogous)
  For simplicity assume $\hat{b}, s$ to be affine on each triangle (otherwise use numerical quadrature)



affine transformation

$$\Phi_k : x \mapsto \hat{x}_{i_{k1}} + \underbrace{(\hat{x}_{i_{k2}} - \hat{x}_{i_{k1}} \mid \hat{x}_{i_{k3}} - \hat{x}_{i_{k1}})}_{=:A_k} x$$

FE basis fcns $\varphi_{i_{k1}}^j, \varphi_{i_{k2}}^j, \varphi_{i_{k3}}^j$

deformed basis fcns:

$$\varphi_{i_{k1}}^j \circ \Phi_k = e_j \varphi_1 \qquad \varphi_1 = 1 - x_1 - x_2$$
$$\varphi_{i_{k2}}^j \circ \Phi_k = e_j \varphi_2 \qquad \text{with} \quad \varphi_2 = x_1$$
$$\varphi_{i_{k3}}^j \circ \Phi_k = e_j \varphi_3 \qquad \varphi_3 = x_2$$

$$\int_{T_k} \hat{b}\cdot\varphi_{i_{kn}}^j \, dx = \int_T \hat{b}\circ\Phi_k \cdot \varphi_{i_{kn}}^j \circ \Phi_k \, |\det D\Phi_k| \, dx = \sum_{l=1}^3 \hat{b}_j(\hat{x}_{i_{kl}}) \int_T \varphi_l \varphi_n \, dx \, |\det A| = \frac{\det A}{24}\left[\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} \hat{b}(\hat{x}_{i_{k1}}) \\ \hat{b}(\hat{x}_{i_{k2}}) \\ \hat{b}(\hat{x}_{i_{k3}}) \end{pmatrix}\right]_n$$

$$\left(\int_{T_k} C\varepsilon(\varphi_{i_{km}}^j):\varepsilon(\varphi_{i_{kn}}^o) \, dx\right)_{mn} = \left(\int_{T_k} C D\varphi_{i_{km}}^j : D\varphi_{i_{kn}}^o \, dx\right)_{mn} = \left(\int_T C(e_j D\varphi_m A^{-1}):(e_o D\varphi_n A^{-1}) \det A \, dx\right)_{mn}$$

$$= \left(\int_T D\varphi_m A^{-1} (C_{mjno})_{jo} A^{-T} D\varphi_n^T \det A \, dx\right)_{mn} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\left[\frac{\det A}{2} A^{-1} (C_{mjno})_{jo} A^{-T}\right]\begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

assembly: for all triangles T_k

compute $\left(\int_{T_k} \hat{b}\cdot\varphi_{i_{kn}}^j\right)_{nj} =: \tilde{B}_{nj}$ and $\left(\int_{T_k} C\varepsilon(\varphi_{i_{km}}^j):\varepsilon(\varphi_{i_{kn}}^o) \, dx\right)_{mn} =: L_k^{jo}$

add mn-entry of L_k^{jo} into L at row i_{km} and column i_{kn}

add $\tilde{B}_{nj}$ into B at row belonging to $\varphi_{i_{kn}}^j$