$$\Rightarrow dS^* = L(t_1,q_1,\dot q_1)\,dt_1 + \int_{t_0}^{t_1}\left(\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right)\right)\delta q\,dt + \frac{\partial L}{\partial \dot q}(t_1,q_1,\dot q_1)\,\delta q(t_1)$$

$$= \left(L(t_1,q_1,\dot q_1) - p_1\dot q_1\right)dt_1 = -H(t_1,q_1,p_1)\,dt_1$$

Thus $\frac{\partial S^*}{\partial t_1} = -H(t_1,q_1,p_1)$ & $\frac{\partial S^*}{\partial q_1} = p_1$. Inserting 2nd into 1st eqn,

$$\frac{\partial S^*}{\partial t_1} + H\left(t_1, q_1, \frac{\partial S^*}{\partial q_1}\right) = 0,$$

hence u before had the interpretation of the value of the extremalized action.

This connects to optimization problems in optimal control, finance, dynamic programming:
A typical pb is given by

$$\min_{\substack{\dot x = f(x,z)\\ x(t_0)=x_0}} \underbrace{\int_{t_0}^{T} h(x(t),z(t))\,dt}_{\text{"running costs"}} + \underbrace{F(x(T))}_{\text{"final cost"}}$$

where $z(t)$ is the "control" (e.g. a price we can set or a physical parameter in a technical process) and x the "state", satisfying the "state eq." $\dot x = f(x,z)$ (x could e.g. be number of consumers ...). The final time T is fixed.

Call the minimum $u(t_0,x_0)$; we try to find u for all t_0 and x_0 simultaneously by deriving a corresponding PDE for u. By the "dynamic programming principle" or "Bellman optimality principle",

$$u(t_0,x_0) = \min_z\left(u(t_0+dt, x_0+dx) + h(x_0,z)\,dt\right)$$

$$\approx \min_z\left(u_t\,dt + u_x\cdot f(x_0,z)\,dt + h(x_0,z)\,dt + u(t_0,x_0)\right)$$

$$= u(t_0,x_0) + dt\left[u_t + \min_z\left(f(x_0,z)\cdot u_x + h(x_0,z)\right)\right],$$

thus $\boxed{u_t + \min_z\left(f(x,z)\cdot u_x + h(x,z)\right) = 0.}$

This is the Hamilton-Jacobi-Bellman equation. Its initial conditions are $u(T,x) = F(x)$.
In the case of minimizing the action, we have $f(x,z) = z$ and $h = L$, thus writing q for x,

$$0 = u_t + \min_z\left(z\cdot u_q + L(q,z)\right) = u_t - H(q,-u_q).$$

Here, u is the action where we keep t_1 and q_1 fixed this time, i.e.

$$u(t_0,q_0) = \min_{\substack{q(t_1)\text{ fixed \& given}\\ q(t_0)=q_0}} S[q].$$

Making the extremalized action a function of t_0 and q_0 instead of t_1 and q_1 is the reason for the two additional minus-signs in the HJB eqn, which we did not have for S^*.

Solutions of HJB eqns are typically non-classic ⇒ leads to notion of "viscosity solutions".

Link to geometry

- Geodesics on hypersurfaces $\tilde S \subset \mathbb{R}^n$ are paths $x(t)\in\tilde S$ minimizing path length $\mathcal{L}$ for fixed endpoints $x(t_0), x(t_1)$, where

$$\mathcal{L}[x] = \int_{t_0}^{t_1}|\dot x|\,dt$$

- assoc. geometric variational pb: $\min_{\substack{x(t_0)=x_0\\ x(t_1)=x_1}} \mathcal{L}[x]$

let $\delta x(t)$ be perturbation of $x(t)$ with $\delta x(t_0) = \delta x(t_1) = 0$ and δx tangent to $\tilde S$

$$\Rightarrow 0 = \delta\mathcal{L} = \int_{t_0}^{t_1}\frac{\dot x}{|\dot x|}\cdot\delta\dot x\,dt = -\int_{t_0}^{t_1}\frac{d}{dt}\left(\frac{\dot x}{|\dot x|}\right)\cdot\delta x\,dt$$

$\Rightarrow \frac{d}{dt}\left(\frac{\dot x}{|\dot x|}\right)\perp\tilde S$ at all times

- corresp. mechanical variational pb.: masspoint constrained to travel on $\tilde S$, but no other forces
action $S = \int_{t_0}^{t_1}\frac{1}{2}|\dot x|^2\,dt$ is minimized ⇒ for any admissible perturbation δx,

$$0 = \delta S = \int_{t_0}^{t_1}\dot x\cdot\delta\dot x\,dt = -\int_{t_0}^{t_1}\ddot x\cdot\delta x\,dt \quad\Rightarrow \ddot x\perp\tilde S \text{ at all times}$$

• connection: the mass point has constant speed and traverses the geodesic!

$\Rightarrow$ one-to-one relation between geometric & mechanical pb.

Now consider a manifold $\tilde{S}$ with Riemannian metric $g_x(\xi, \eta) = \xi^T A(x)\eta$

• geometric pb.: $\min_{\substack{x(t_0)=x_0 \\ x(t_1)=x_1}} \mathcal{L}[x]$ with $\mathcal{L}[x] = \int_{t_0}^{t_1} \sqrt{\dot{x}^T A(x)\dot{x}}\, dt$

• mechanical pb: $\min_{\substack{x(t_0)=x_0 \\ x(t_1)=x_1}} S[x]$ with $S[x] = \int_{t_0}^{t_1} \frac{1}{2}\dot{x}^T A(x)\dot{x}\, dt$ (mass point on $\tilde{S}$ with kinetic energy $\frac{1}{2}\dot{x}^T A(x)\dot{x}$ and no potential energy)

• connection: mass point has constant $\sqrt{\dot{x} A(x)\dot{x}}$ and traverses geodesic

Indeed, let x be extremal for mech. pb.

$$\Rightarrow -\frac{d}{dt}(A(x)\dot{x}) + \frac{1}{2}\dot{x}^T DA(x)\dot{x} = 0$$

$$\Rightarrow \frac{d}{dt}\left(\frac{A(x)\dot{x}}{\sqrt{\dot{x}^T A(x)\dot{x}}}\right) = \frac{1}{2}\frac{\dot{x}^T DA(x)\dot{x}}{\sqrt{\dot{x}^T A(x)\dot{x}}}$$, since $\sqrt{\dot{x}^T A(x)\dot{x}} = \text{const}$; indeed $T = \frac{1}{2}\dot{x}^T A(x)\dot{x}$

$\Rightarrow H = \frac{1}{2} p^T A^{-1}(x) p = \frac{1}{2}\dot{x}^T A(x)\dot{x} = \text{const.}$

$\Rightarrow$ x is extremal for $\mathcal{L}$ and has constant $\sqrt{\dot{x}^T A(x)\dot{x}}$

Now let x be a geodesic

$$\Rightarrow \frac{d}{dt}\left(\frac{A(x)\dot{x}}{\sqrt{\dot{x}^T A(x)\dot{x}}}\right) = \frac{1}{2}\frac{\dot{x}^T DA(x)\dot{x}}{\sqrt{\dot{x}^T A(x)\dot{x}}}$$

$\Rightarrow$ choose parametrization with $\dot{x}^T A(x)\dot{x} = \text{const.}$ $\Rightarrow$ x extremal for action.

Now what about a mechanical system with potential?

I.e. $L = T - V = \frac{1}{2}|\dot{x}|^2 - V(x)$, $H = \frac{1}{2}|\dot{x}|^2 + V(x) = \text{const.}$

Here, as before, one can show that an extremal path of the action with energy $H = E$ is also extremal for $\mathcal{L}[x] = \int_{t_0}^{t_1} \sqrt{2(E - V(x))}\, |\dot{x}|\, dt$ and vice versa (up to reparametrization)