- <u>augment action</u> $S[(r,\theta)]$ by a Lagrange multiplier term $\underbrace{(\lambda, \underbrace{r-\ell}_{\text{constraint}})}_{\text{dual pairing}} = \int_0^\tau \lambda(r-\ell)\,dt$, λ: Lagrange multiplier

$$\hat{S}[(r,\theta),\lambda] = \int_0^\tau \frac{m}{2}(\dot r^2 + r^2\dot\theta^2) + gmr\cos\theta - \lambda(r-\ell)\,dt$$

(in finite dimensional optimization, one has one Lagrange multiplier for each constraint; here we have infinitely many constraints, i.e. $r(t)=\ell$ for every t, thus also a value of λ at each time t)

- a <u>saddle point</u> of $\hat S$ is a critical path among all from $q(0)$ to $q(\tau)$ which satisfy the constraint;

Opt. / saddle point conditions:

$$0 = \delta_r \hat S(\vartheta) = \frac{d}{d\rho}\hat S[(r+\rho\vartheta,\theta),\lambda]\Big|_{\rho=0} = \int_0^\tau m\dot r\dot\vartheta + mr\vartheta\dot\theta^2 + gm\vartheta\cos\theta - \lambda\vartheta\,dt \underset{\vartheta(0)=\vartheta(\tau)=0}{=} \int_0^\tau \vartheta(-m\ddot r + mr\dot\theta^2 + gm\cos\theta - \lambda)\,dt$$

$$\Rightarrow \ddot r = r\dot\theta^2 + g\cos\theta - \frac{\lambda}{m} \qquad (a)$$

$$0 = \delta_\theta \hat S(\vartheta) = \int_0^\tau \vartheta\left(-m\frac{d}{dt}(r^2\dot\theta) - gmr\sin\theta\right)dt \Rightarrow \frac{d}{dt}(r^2\dot\theta) + gr\sin\theta = 0 \qquad (b)$$

$$0 = \delta_\lambda \hat S(\vartheta) \Rightarrow r = \ell \qquad (c)$$

$\Rightarrow$ three coupled equations (here, by substituting (c) into (a), (b), $\ddot\theta = -\frac{g}{\ell}\sin\theta$, $\lambda = m(\ell\dot\theta^2 + g\cos\theta)$)

- Note: The Lagrange multiplier in general represents the <u>constraint forces</u> F_c in the direction of the constrained variable. Indeed, the unconstrained Newton's law is

$$m\underbrace{\left[(\ddot r - r\dot\theta^2)\begin{pmatrix}\sin\theta\\ -\cos\theta\end{pmatrix} + (2\dot r\dot\theta + r\ddot\theta)\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}\right]}_{\text{acceleration}} = \begin{pmatrix}0\\ -mg\end{pmatrix} + F_c$$

which is $\Leftrightarrow$ to (a), (b), (c) for $F_c = m(\ell\dot\theta^2 + g\cos\theta)\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}$

<u>Questions</u>: · If action has certain symmetries, what symmetries are conserved in augmented action?
· What does this imply about energy / momentum conservation?

<u>Summary: advantages of Lagrangian perspective</u>

- coordinate invariance $\rightsquigarrow$ choose simplest coordinate system
- simple incorporation of constraints
- variational approach $\rightsquigarrow$ amenable to variational techniques (e.g. existence proof or variational discretization)
- one-to-one relation between symmetries and first integrals (Noether's theorem; finding such symmetries is challenging $\rightsquigarrow$ canonical transformation theory)

<u>Hamiltonian perspective</u> (recall $\dot q = \frac{\partial H}{\partial p}$, $\dot p = -\frac{\partial H}{\partial q}$ $(***)$)

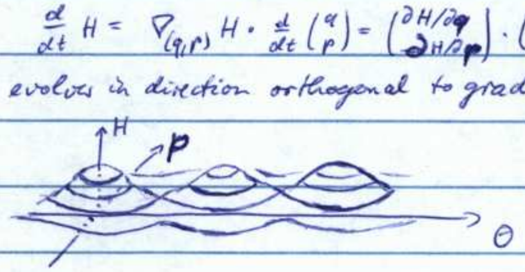
<u>phase space interpretation</u>

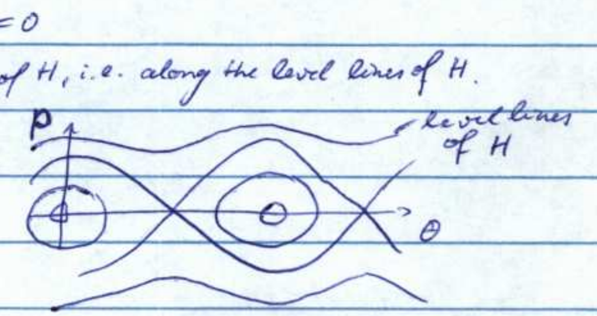
Assume H does not depend on time; in which case we have shown energy conservation, $\frac{d}{dt}H = 0$.
This energy conservation can be illustrated quite intuitively: The $2N$ coordinate dimensions $q_1, \ldots, q_N, p_1, \ldots, p_N$ make up phase space. Now

$$\frac{d}{dt}H = \nabla_{(q,p)}H \cdot \frac{d}{dt}\begin{pmatrix}q\\ p\end{pmatrix} = \begin{pmatrix}\partial H/\partial q\\ \partial H/\partial p\end{pmatrix}\cdot\begin{pmatrix}\dot q\\ \dot p\end{pmatrix} = 0$$

$\Rightarrow$ System evolves in direction orthogonal to gradient of H, i.e. along the level lines of H.

<u>Ex:</u> P)





<u>Liouville's Theorem</u>

<u>Def. (Phase flow)</u>: The phase flow is the one-parameter group of transformations of phase space

$$g^t : (q(0), p(0)) \mapsto (q(t), p(t))$$

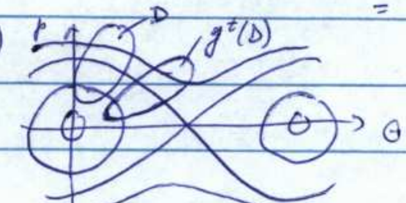
for $q(t), p(t)$ solution of $(***)$.

Thm (Liouville): The phase flow preserves volume, i.e. for any measurable set $D \in \mathbb{R}^{2N}$,
$$vol(g^t(D)) = vol(D).$$

proof: We will show $\frac{d}{dt} vol(g^t(D))\big|_{t=0} = \int_D \operatorname{div} f(x)\,dx$ (1) for any flow g^t and any vector field f satisfying $g^t(x) = x + f(x)t + O(t^2)$ as $t \to 0$. In our case, $x = \binom{q}{p}$, $f(q,p) = \begin{pmatrix} \partial H/\partial p \\ -\partial H/\partial q \end{pmatrix}$, and thus $\operatorname{div}_{(q,p)} f = \frac{\partial}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial}{\partial p}\frac{\partial H}{\partial q} = 0$ so that $\frac{d}{dt} vol(g^t(D))\big|_{t=0} = 0$, but then also
$\frac{d}{dt} vol(g^t(D))\big|_{t=\tau} = \frac{d}{dt} vol(g^t(g^\tau(D)))\big|_{t=0} = \int_{g^\tau(D)} \operatorname{div} f \, d(q,p) = 0$ for all times τ.

Now show (1): $vol(g^t(D)) = \int_D \det Dg^t\, dx = \int_D \det(I + tDf + O(t^2))\,dx = \int_D 1 + t \operatorname{tr} Df + O(t^2)\,dx$
$= vol(D) + t \int_D \operatorname{div} f\, dx + O(t^2)$. □

Ex: P)



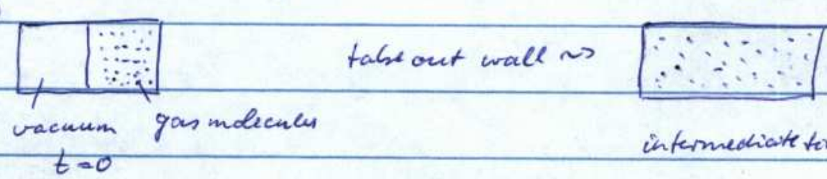
D and $g^t(D)$ have same area!

Liouville's thm implies that a Hamiltonian system cannot have asymptotically stable equil. states or limit cycles. It is the basis of applying ergodic theory to mechanics (see also stat. mech. part); an example is the following:

Thm (Poincaré recurrence thm): Let g be a volume-preserving continuous one-to-one mapping which maps a bdd region D of Euclidean space onto itself, $g(D) = D$. Then in any neighborhood $U \subset D$ of any point in D there is a point $x \in U$ which returns to U, i.e. $g^n(x) \in U$ for some $n > 0$.

proof: If $U, g(U), g^2(U), \ldots$ never intersected, D would have infinite volume, thus
$g^k(U) \cap g^l(U) \neq \emptyset$ or $g^n(U) \cap U \neq \emptyset$ for $n = k - l$. For $y \in g^n(U) \cap U$, also $x = g^{-n}(y) \in U$. □

Ex: MB)



⇝ at some time we get arbitrarily close to initial state!

The Poincaré recurrence thm can be employed e.g. if from energy conservation we obtain boundedness of q, p. Then we define $D = \{(q,p) \in \mathbb{R}^{2N} \mid H(q,p) \in E\}$.

Canonical action principle

One can define the canonical action $S[q,p] = \int_0^\tau p \cdot \dot q - H(t,q,p)\,dt$, which now depends on q and p, which can both be chosen freely (unlike q and $\dot q$ in the original action $S[q] = \int_0^\tau L(t,q,\dot q)\,dt$). The original action is recovered as $S[q] = \max_{p:[0,\tau]\to\mathbb{R}^N} S[q,p]$, hence the dynamics of a mechanical system are obtained by requiring that the pair (q,p) is extremal wrt canonical action (given $q(0)$ and $q(\tau)$). This time, the Euler-Lagrange equations are first order and given by (∗∗∗).

Summary: advantage of Hamiltonian perspective
- volume-preserving in phase space (Liouville) ⇝ ergodic theory applicable
- first order system

Link to Hamilton-Jacobi (HJ) equations

Consider 1st order PDE in u,
$$u_t + H(\nabla u, x) = 0 \qquad (1)$$
for some H. This PDE can be solved by instead solving the system of ODEs for $x(t)$, $p(t) = \nabla u(t)$, $u(t)$
$$\frac{dx}{dt} = \nabla_p H, \quad \frac{dp}{dt} = -\nabla_x H, \quad \frac{du}{dt}(x(t),t) = p \cdot \dot x - H(p,x) \underset{\dot x = \nabla_p H}{=} L(x, \dot x), \qquad (2)$$

Before showing the relation between (1) & (2), note that (2) is remarkable: the ODEs for x and p (so-called "characteristics") are exactly Hamilton's equations ($x \leftrightarrow x$), and along the corresponding paths, u changes according to the Lagrangian! What is u?

Now, why does (2) hold:

- Differentiating (1) $\Rightarrow \dfrac{\partial^2 u}{\partial x_i \partial t} + \sum_j \dfrac{\partial H}{\partial p_j} \dfrac{\partial^2 u}{\partial x_j \partial x_i} + \dfrac{\partial H}{\partial x_i} = 0$
- also, for any curve $x(t)$, $\dfrac{d}{dt} \nabla_{x_i} u(x(t),t) = \dfrac{\partial^2 u}{\partial x_i \partial t} + \sum_{j=1}^{N} \dfrac{\partial^2 u}{\partial x_i \partial x_j} \dfrac{dx_j}{dt}$
- now choose curve $\dot{x} = \nabla_p H$ and add both eqs. $\Rightarrow \dot{p} = \frac{d}{dt} \nabla u = -\nabla_x H$
- finally, $\dfrac{du}{dt}(x(t),t) = \nabla_x u \cdot \dot{x} + \dfrac{\partial u}{\partial t} = p \cdot \dot{x} - H(p,x)$.

Method of Characteristics

Replacing (1) by (2) is a special case of the method of characteristics. This method can be used to solve PDEs of the form

$$F(t, x, u_t, u_x, u) = 0.$$

Consider a path $t(\tau), x(\tau)$ on the t-x-plane, then u, u_t, u_x are fcns of τ with

$$u' = u_t t' + u_x x', \quad u_t' = u_{tt} t' + u_{tx} x', \quad u_x' = u_{xt} t' + u_{xx} x'$$

Problem: $u_{tt}, u_{xt} = u_{tx}, u_{xx}$ are unknown

Remedy: We have the eqns

$$0 = \frac{d}{dt} F = F_t + F_{u_t} u_{tt} + F_{u_x} u_{xt} + F_u u_t \qquad (a)$$

$$0 = \frac{d}{dx} F = F_x + F_{u_t} u_{tx} + F_{u_x} u_{xx} + F_u u_x \qquad (b)$$

However, still missing one eq. to be able to solve for u_{tt}, u_{tx}, u_{xx} in terms of u, u_t, u_x. But we may still choose $t(\tau), x(\tau)$ in a special way so that terms cancel.

Choose

$$t' = \frac{\partial F}{\partial u_t}, \quad x' = \frac{\partial F}{\partial u_x} \qquad (c)$$

then

$$u' = u_t \frac{\partial F}{\partial u_t} + u_x \frac{\partial F}{\partial u_x}$$

$$u_t' = u_{tt} \frac{\partial F}{\partial u_t} + u_{tx} \frac{\partial F}{\partial u_x} \overset{(a)}{=} -\frac{\partial F}{\partial t} - u_t \frac{\partial F}{\partial u}$$

$$u_x' = u_{xt} \frac{\partial F}{\partial u_t} + u_{xx} \frac{\partial F}{\partial u_x} \overset{(b)}{=} -\frac{\partial F}{\partial x} - u_x \frac{\partial F}{\partial u}$$

The curves defined by (c) are called characteristics; instead of solving the PDE we can solve the above ODEs along the characteristics (the initial value of the ODEs is given by boundary conditions for PDE). As long as the curves (c) do not cross, the method is feasible; at crossings shocks typically form which can then be treated using special techniques ("Rankine-Hugoniot"-conditions). The PDE $u_t + H(\nabla u, x) = 0$ is a special case with

$$t' = 1, \quad x' = \nabla_p H, \quad u' = u_t + p \cdot \nabla_p H, \quad u_t' = 0, \quad u_x' = -\nabla_x H$$

Hamilton-Jacobi equations

~~What is the link between mechanics and the PDE $u_t + H(\nabla u, x) = 0$?~~

Consider the action $S[q] = \displaystyle\int_{t_0}^{t_1} L(t, q, \dot{q})\, dt$

Fixing $q(t_0)$, any other state $q(t_1)$ is reached by the mechanical system via minimizing $S[q]$ (or more generally by an extremal path). Let

$$S^*[t_1, q_1] = \min_{\substack{q(t_0)\text{ fixed \& given} \\ q(t_1) = q_1}} S[q]$$

How does S^* vary with t_1, q_1? Change extremal path by δq with $\delta q(t_0) = 0$

$$\Rightarrow dS^* = \int_{t_0}^{t_1} \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q}\, dt = \frac{\partial L}{\partial \dot{q}}(t_1, q_1, \dot{q}_1)\, \delta q(t_1) + \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) \right) \delta q\, dt \underset{q \text{ is extremal}}{=} \frac{\partial L}{\partial \dot{q}}(t_1, q_1, \dot{q}_1)\, \delta q(t_1)$$

$$\Rightarrow dS^* = p(t_1)\, dq_1$$

Now extend the path by time dt_1 and perturb it by δq with $\delta q(t_0) = 0$ and

$$q(t_1) = q(t_1 + dt_1) + \delta q(t_1 + dt_1) \underset{\text{to first order}}{\Longrightarrow} \delta q(t_1) = -\dot{q}_1\, dt_1$$